Scientific Bulletin of the Petru Maior University of Tirgu Mures
Vol. 6 (XXIII), 2009
ISSN 1841-9267

Expansion of the notions of normed algebras to locally multiplicatively convex algebras

Diana Mărginean
"Petru Maior" University of Tirgu Mures
N. Iorga street nr. 1, 540088, Romania
dianapetrovai@yahoo.com

Nicola Oprea
"Petru Maior" University of Tirgu Mures
N. Iorga street nr. 1, 540088, Romania
noprea@engineering.upm.ro

Abstract

We generalize many of the theories of numerical range for the normed algebras to a theory of locally m-convex algebras. We will establish relations approximation between numerical range and spectrum of the element from locally m-convex unitary complete algebra.

1 Introduction

In a Banach algebra the spectrum of an element does not depend on the norm, but only on the algebric structure.

The numerical range of an element in a normed algebra is a subset of the scalar field which together reflect the normed and algebric structure.

When the algebra is complete, the numerical range of an element contains the spectrum of this element.

In a unitary normed algebra the numerical radius is a norm which is equivalent with the given norm.

We expand the notions of normed algebras to locally m-convex algebras. It is shown the approximate relations between the numerical range and the spectrum of an element are maintained in the generalization.

For a normed complex unitary algebra $(A, \|\cdot\|)$ we define the set: $D(A, \|\cdot\|; 1) = \{f \in A' | f(1) = 1 \text{ and } \|f\| = 1\}$. For any $a \in A$ we define numerical range of a the set

$$V(A, \|\cdot\|; a) = \{f(a) | f \in D(A, \|\cdot\|; 1)\},$$

and numerical radius the set:

$$v(A, \|\cdot\|; a) = \sup\{|\lambda| \mid \lambda \in V(A, \|\cdot\|; 1)\}.$$

The set $D(A, \|\cdot\|; 1)$ is a convex subset, weak compact of A' and numerical range $V(A, \|\cdot\|; a)$ is also a compact subset of **C**, [2].

The properties and applications of numerical ranges on a normed algebra have been largely studied and the main results have been presented by F.F. Bonsall and J.Duncan [2]. The m-convex locally algebras have been thoroughly examined by E.A. Michael in [5].

We want to expand the concept of numerical range from normed unitary complex algebras to locally m-convex algebras. For this it is sufficient to observe that for a given m-convex locally algebra A, with unital 1 there exists an increasingly family of submultiplicatively seminormes $\{p_\alpha\}$ on A which generates the topology such that $p_\alpha(1) = 1$ for all α. Given this algebra we denote with $P(A)$ the class of all these family of seminormes on A and with $(A, \{p_\alpha\})$ the algebra A with the family $\{p_\alpha\}$ fixed by seminormes $\{p_\alpha\} \in P(A)$.

Given $(A, \{p_\alpha\})$ for each α we denote with N_α the null subspace of p_α, through A_α factor subspace $A|_{N_\alpha}$ and with $\|\cdot\|_\alpha$ we denote the norm on A_α, defined by $\|x + N_\alpha\|_\alpha = p_\alpha(x)$. For each α, we consider the linear canonical map $x \mapsto x_\alpha \equiv x + N_\alpha$ from A to A_α. We denote by 1_α the unital element in A_α and it results that $\|1_\alpha\|_\alpha = 1$ for all α. Michael has obtained the significant result that A is isomorph with a subalgebra of the product of normed algebras $(A_\alpha, \|\cdot\|_\alpha)$.

Using this characterization of locally m-convex algebras we generalize many of the theories of numerical range for the normed algebras to a theory of locally m-convex algebras.

2 Numerical range

Given $(A, \{p_\alpha\})$, we define the set: $D_\alpha(A, p_\alpha; 1) \equiv \{f \in A' | f(1) = 1 \text{ and } \|f(x)\| \le p_\alpha(x), \text{ for all } x \in A\}$, and write

$$D(A, p_\alpha; 1) = \bigcup_\alpha \{D_\alpha(A, p_\alpha; 1)\}.$$

For all $a \in A$ we write

$$V_\alpha(A, p_\alpha; a) \equiv \{f(a) | f \in D_\alpha(A, p_\alpha; 1)\}$$

Keywords: *theory of locally m-convex algebras, spectrum and numerical range*

and define numerical range as the set

$$V(A,\{p_\alpha\};a)\equiv\bigcup_\alpha\{V_\alpha(A,p_\alpha;a)\}.$$

To each linear functional f on (A,p_α) which becomes null on N_α we can associate the linear functional F on A_α through $F(x_\alpha)=f(x)$ and to each F pe A_α we can associate the linear functional f on (A,p_α) trough $f(x)=F(x_\alpha)$.

Hence it follows from the definition of the norm on A_α that $D_\alpha(A,p_\alpha;1)$ is isomorphic with $D(A_\alpha,\|\cdot\|_\alpha;1_\alpha)$ and for $a\in A$ we have:

$$V_\alpha(A,p_\alpha,a)=V(A_\alpha,\|\cdot\|_\alpha;a_\alpha).$$

Hence it results that numerical range of a is characterized by numerical ranges of a_α in normed algebras such as:

$$V(A,\{p_\alpha\};a)=\bigcup_\alpha\{V(A_\alpha,\|\cdot\|_\alpha;a_\alpha)\}.$$

Both $D(A,\{p_\alpha\};1)$ and $V(A,\{p_\alpha\};a)$ depend on the chosen fixed family of seminorms associated to algebra A. It is obviously that when $\{p_\alpha\}$ is an increasingly family of seminorms, $D(A,\{p_\alpha\};1)$ is a convex subset of A' and numerical range $V(A,\{p_\alpha\};a)$ is a convex subset of $\mathbf{C}$.

For any $a\in A$ we write:

$$v_\alpha(A,p_\alpha;a)\equiv\sup\{|\lambda|\mid\lambda\in V_\alpha(A,p_\alpha;a)\}$$

and we define numerical radius of a as:

$$v(A,\{p_\alpha\};a)\equiv\sup\{|\lambda|\;|\lambda\in V(A,p_\alpha;a)\}.$$

We have:

$$v_\alpha(A,p_\alpha;a)\le p_\alpha(a)$$

for all α and putting $v(A,\{p_\alpha\};a)=\infty$, we have that:

$$v(A,\{p_\alpha\};a)=\sup_\alpha v_\alpha(A,p_\alpha;a)=\sup_\alpha v_\alpha(A,\|\cdot\|_\alpha;a).$$

It is obvious that numerical range and numerical radius have the following properties for $a\in A$ and λ,μ complex:

$$V(A,\{p_\alpha\};\lambda a+\mu\cdot 1)=\lambda V(A,\{p_\alpha\};a)+\mu$$

and

$$v(A,\{p_\alpha\};\lambda a+\mu\cdot 1)\le|\lambda|v(A,\{p_\alpha\}_\alpha;a)+|\mu|.$$

and for $a,b\in A$ we have:

$$V(A,\{p_\alpha\};a+b)\subseteq V(A,\{p_\alpha\};a)+V(A,\{p_\alpha\};b)$$

and

$$v(A,\{p_\alpha\};a+b)\le v(A,\{p_\alpha\};a)+v(A,\{p_\alpha\};b).$$

2.1 The spectrum and numerical range of the element

We now establish approximation relations between spectrum and numerical range of an element.

We know that given the unitary algebra A, for any $a\in A$, spectrum of a is defined as:

$$\sigma(A;a)\equiv\{\lambda|a-\lambda\cdot 1\text{is non-invertible}\}.$$

Theorem 1. *Let A be an unitary locally m-convex algebra and $x\in A$. Then*

$$\sigma(A;x)\neq\emptyset.$$

Proof. If $x\notin G(A)$ it follows that $0\in\sigma(A;x)$ and so

$$\sigma(A;x)\neq\emptyset.$$

Let $x\in G(A)$. We suppose that $\sigma(A;x)=\emptyset$. Hence $C\sigma(A;x)=\mathbf{C}$. It follows that for all $\lambda\in\mathbf{C}$, $\lambda\cdot 1-x\in G(A)$. Hence $\lambda(1-\frac{1}{\lambda}x)\in G(A)$, for all $\lambda\neq 0$.
It follows that for all $z\in\mathbf{C}^*$ we have $1-zx\in G(A)$.
Since $1\in G(A)$ it follows that $1-zx\in G(A)$ for all $z\in\mathbf{C}$.
We have from corollary [A.2, 9], taking $I=0$ and $p_\alpha(1-0)\ge 1$, for all α, that exists $f\in A'$ such that:
$f|_I=0, f(1)=1, |f(x)|\le p_\alpha(x)$, for all $x\in A$. Let be the map $F:\mathbf{C}\to\mathbf{C}$, defined by:

$$\begin{aligned}&F(z)=f((1-zx)^{-1}).\\&\frac{F(z)-F(z_0)}{z-z_0}=\frac{f((1-zx)^{-1}-(1-z_0x)^{-1})}{z-z_0}=\\&=\frac{f((1-zx)^{-1}(z-z_0)x(1-z_0x)^{-1}}{z-z_0})=\\&=f((1-zx)^{-1}(1-z_0)^{-1})x.\end{aligned}$$

(Since $(1-zx)x=x(1-zx)$ we have:

$$\begin{aligned}x&=(1-zx)^{-1}x(1-zx)\Rightarrow x(1-zx)^{-1}=\\&=(1-zx)^{-1}x).\end{aligned}$$

$$\lim_{z\to z_0}\frac{F(z)-F(z_0)}{z-z_0}=f((1-z_0)^{-2}x)=F'(z_0).$$

Since F is $\mathbf{C}$ - derivable results that F is holomorphic. We have:

$$\begin{aligned}\lim_{z\to\infty}F(z)&=\lim_{z\to\infty}f([z(\frac{1}{z}\cdot 1-x)]^{-1})=\\&=\lim_{z\to\infty}\frac{1}{z}f((\frac{1}{z}\cdot 1-x)^{-1})=0.\end{aligned}$$

Hence if follows that F is constant, $F(0)=F(z)=1$. But

$$\begin{aligned}f((1-zx)^{-1})&\le p_\alpha((1-zx)^{-1})=\\&=\frac{1}{|z|}p_\alpha((\frac{1}{z}1-x)^{-1})\to_{z\to\infty}0.\end{aligned}$$

It follows that $1\le 0$ is contradiction. Hence $\sigma(A;x)\neq\emptyset$. □

Proposition 1. *Let be A a complete locally m-convex algebra and $x \in A$. If A is a q-algebra (i.e. $G^q(A)$ - open set), then $\sigma(A;x)$ is a compact set.*

Proof. Since $G^q(A)$ is open set and $0 \in G^q(A)$, it follows that there exists an equilibrated neighborhood V of 0, such that $V \subseteq G^q(A)$. Since V is absorbent it follows that there exists $\alpha > 0$ such that $\alpha x \in V$.
Let $\rho(x) := \sup_{\lambda \in \sigma(A;x)} |\lambda|$, $\rho(x)$ is call spectral radius of x.

We show that $\rho(x) \leq \frac{1}{\alpha}$.

Suppose by absurd $\rho(x) > \frac{1}{\alpha}$.

It follows that there exists $\lambda \in \sigma(A;x)$, such that $|\lambda| > \frac{1}{\alpha}$.

It follows that $|\frac{1}{\alpha\lambda}| < 1$. Since V is equilibrated, then $\frac{1}{\alpha\lambda}\alpha x \in V$.

It follows that there exists $\frac{1}{\lambda}x \in V$, so $\frac{1}{\lambda}x \in G^q(A)$, so $\lambda \notin \sigma(A;x)$, contradiction.

Therefore $\rho(x) \leq \frac{1}{\alpha}$, hence $\sigma(A;x)$ is bounded.

Let $C\sigma(A;x) = \{\lambda \in \mathbf{C} | \lambda \cdot 1 - x \in G(A)\}$.

Let $\lambda_0 \in C\sigma(A;x)$, then $\lambda_0 \cdot 1 - x \in G(A)$, it follows that there exists $p_{\alpha_1}, ..., p_{\alpha_n}$ such that:

$$B_{p_{\alpha_1},...,p_{\alpha_n}}(\lambda_0 \cdot 1 - x, r) \subseteq G(A).$$

Therefore $p_{\alpha_i}(\lambda \cdot 1 - x - (\lambda_0 \cdot 1 - x)) \leq r$, for all $i = \overline{1,n}$ if and only if $|\lambda - \lambda_0| < r$. Hence $B(\lambda_0, r) \subseteq C\sigma(A;x)$, it follows that $C\sigma(A;x)$ is open set, since $\sigma(A;x)$ is a closed set.

Since $\sigma(A;x)$ is bounded and closed set it follows $\sigma(A;x)$ is a compact set. □

Proposition 2. *Let be A a locally m-convex, unitary algebra. We denote by*

$$G_x := \{\lambda \in \mathbf{C} | 1 - \lambda x \in G(A)\}$$

and we suppose there exits $r > 0$ such that $\overline{D_r(0)} \subseteq G_x$. Let $f : G_x \to A, f(\lambda) = (1 - \lambda x)^{-1}$.
Then:
1) f is derivable on $D_r(o)$ and $f'(\lambda) = (1 - \lambda x)^{-2}x$.
2) f^m is indefinitely derivable on $D_r(o)$ and $(f^m)^{(n)}(\lambda) = m(m+1)...(m+n-1)f^{m+n}(\lambda)x^n$, for all $m \in \mathbf{N}^*, n \in \mathbf{N}$.
3) For any $s \in A'$, $s(f^m(\lambda))^{(n)} = m(m+1)...(m+n-1)s(f^{m+n}(\lambda)x^n)$.

Proof. 1) $\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} =$
$= \lim_{\lambda \to \lambda_0} \frac{(1 - \lambda x)^{-1} - (1 - \lambda_0 x)^{-1}}{\lambda - \lambda_0} =$
$= \lim_{\lambda \to \lambda_0} \frac{(1 - \lambda x)^{-1}(1 - \lambda_0 x)^{-1}(\lambda - \lambda_0)x}{\lambda - \lambda_0} =$
$= (1 - \lambda_0 x)^{-2}x = f^2(\lambda_0)x.$
Hence:

$$f'(\lambda) = f^2(\lambda)x, \text{ for all } \lambda \in D_r(0).$$

$$f''(\lambda) = 2f'(\lambda)x = 2f^3(\lambda)x^2, \text{ for all } \lambda \in D_r(0).$$

2) $f^m(\lambda) = (1 - \lambda x)^{-m}$

$$\lim_{\lambda \to \lambda_0} \frac{f^m(\lambda) - f^m(\lambda_0)}{\lambda - \lambda_0} =$$
$$= \lim_{\lambda \to \lambda_0} \frac{(1 - \lambda x)^{-m} - (1 - \lambda_0 x)^{-m}}{\lambda - \lambda_0} =$$
$$= \lim_{\lambda \to \lambda_0} \frac{(1 - \lambda x)^{-m}[(1 - \lambda x)^{-m} - (1 - \lambda_0 x)^{-m}](1 - \lambda_0 x) - m}{\lambda - \lambda_0} =$$
$$= \lim_{\lambda \to \lambda_0} \left(\frac{\{(\lambda - \lambda_0)x[(1 - \lambda_0 x)^{m-1} + ... + (1 - \lambda x)^{m-1}]\}}{\lambda - \lambda_0} \cdot \right.$$
$$\left. \cdot \frac{(1 - \lambda x)^{-m}(1 - \lambda_0 x)^{-m}}{\lambda - \lambda_0} \right) =$$
$$= m(1 - \lambda_0 x)^{-(m+1)}x, \text{ for all } m \in \mathbf{N}^*$$

Hence $(f^m)'(\lambda) = m(f^{m+1})(\lambda)x$, for all $\lambda \in D_r(0)$.
We show

$$(f^m)^{(n)} = m(m+1)...(m+n-1)f^{m+n}(\lambda)x^n.$$

We have that:

$$(f^m)' = mf^{m+1}(\lambda)x.$$

We suppose that:

$$(f^m)^{(n)} = m(m+1)...(m+n-1)f^{m+n}(\lambda)x^n. \odot$$

We want:

$$(f^m)^{(n+1)} = m(m+1)...(m+n)f^{m+n+1}(\lambda)x^{n+1}.$$

It follows from $\odot$ that:

$$(f^m)^{(n+1)} = m(m+1)...(m+n-1)(m+n) f^{m+n+1}(\lambda)x^{n+1}.$$

through its derivation $\odot$. Therefore fo rall $\lambda \in D_r(0)$

$$(f^m)^{(n)} = m(m+1)...(m+n-1)f^{m+n}(\lambda)x^n,$$

for all $m \in \mathbf{N}^*$, for all $n \in \mathbf{N}$.

3) For any $s \in A'$ we have:

$$\lim_{\lambda \to \lambda_0} \frac{s(f(\lambda)) - s(f(\lambda_0))}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} s(\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0}) = = s(f'(\lambda_0)).$$

Therefore

$$s\left(f^m(\lambda)\right)^{(n)} = m(m+1)...(m+n-1)s(f^{m+n}(\lambda)x^n)$$

for all $\lambda \in D_r(0)$, for all $m \in \mathbf{N}^*$, for all $n \in \mathbf{N}$. □

Theorem 2. *Let be A an unitary, m-convex algebra and we suppose these exists $r > 0$ such that $\overline{D_r(0)} \subseteq G_x, s \in A'$. Then:*
$s(x^n) = \frac{n!m!}{(m+n)!}\frac{1}{2\pi i}\int_{\partial D_\rho(0)} \frac{s(f^{m+1}(\xi))}{\xi^{n+1}}d\xi$ *for all $n \in \mathbf{N}^*$, for all $m \in \mathbf{N}$, where $f : G_x \to A, f(\lambda) = (1 - \lambda x)^{-1}$, for all $0 < \rho < r$.*

Proof. Let be function:
$G_x \to \mathbf{C}$
$\xi \mapsto D(f^{m+1}(\xi))$
derivable on $D_r(0)$ and $\partial D_\rho(0) \subset D_r(0)$, for all $0 < \rho < r$. Then:
$\frac{n!}{2\pi i}\int_{\partial D_\rho(0)} \frac{s(f^{m+1}(\xi))}{(\xi-0)^{n+1}} d\xi = s(f^{m+1}(0))^{(n)} =$
$= (m+1)...(m+1+n-1)s(f^{m+1+n}(0)x^n) = \frac{(m+n)!}{n!} s(x^n)$, for all $0 < \rho < r$.
It follows that: $s(x^n) = \frac{n!m!}{(m+n)!}\frac{1}{2\pi i}\int_{\partial D_\rho(0)} \frac{s(f^{m+1}(\xi))}{\xi^{n+1}} d\xi$, for all $0 < \rho < r$. □

Corollary 1. *Let be A an unitary m-convex algebra, $x \in A$ and there exists $r > 0$ such that $\overline{D_r(0)} \subseteq G_x$. Then $p_\alpha(x^n) \le \frac{n!m!}{(n+m)!}\frac{1}{\rho^n} \sup_{\xi\in\partial D_\rho(0)} p_\alpha(f^{m+1}(\xi))$, for all $0 < \rho < r$.*

Proof. If $p_\alpha(x^n) = 0$, for all α inequality it is evident. We suppose that $\exists\alpha$ such that $p_\alpha(x^n) \neq 0$. We take $y = \frac{x^n}{p_\alpha(x^n)}$, with $p_\alpha(y) = 1$. Let $\mathbf{K} \in \{\mathbf{C}, \mathbf{R}\}, f_1 : \mathbf{K}y \to \mathbf{C}$, $f_1(\lambda y) = \lambda$, $f_1$linear and$|f_1(\lambda y)| = |\lambda| \le |\lambda| p_\alpha(y) = p_\alpha(\lambda y)$. It follows that there exists $s \in A'$ such that: $s|_{\mathbf{K}y} = f_1$, $s(y) = 1$. $|s(y) \le p_\alpha(y)$, for all $y \in A$. Since $s(\frac{x^n}{p_\alpha(x^n)}) = 1$, it follows that $s(x^n) = p_\alpha(x^n)$. since $p_\alpha(x^n) \ge 0$, it follows that $s(x^n) = |s(x^n)|$. Hence $p_\alpha(x^n) = |s(x^n)| =$
$= \frac{n!m!}{(n+m)!}\frac{1}{2\pi i}\left|\int_{\partial D_\rho(0)} \frac{s(f^{m+1}(\xi))}{\xi^{n+1}} d\xi\right| = \frac{n!m!}{(n+m)!}\frac{1}{2\pi i}\left|\int_0^1 \frac{s(f^{m+1}(\rho e^{2\pi i t}))\rho e^{2\pi i t}}{\rho^{n+1} e^{2\pi i t}} dt\right| 2\pi i \le$
$\le \frac{n!m!}{(n+m)!}\frac{1}{\rho^n}\int_0^1 \left|s(f^{m+1}(\rho e^{2\pi i t}))\right| dt \le \frac{n!m!}{(n+m)!}\frac{1}{\rho^n}\int_0^1 \left|p_\alpha(f^{m+1}(\rho e^{2\pi i t}))\right| dt \le$
$\le \frac{n!m!}{(n+m)!}\frac{1}{\rho^n}\sup_{\xi\in\partial D_\rho(0)} p_\alpha(f^{m+1}(\xi))$, for all $0 < \rho < r$.
Hence $p_\alpha(x^n) \le \frac{n!m!}{(n+m)!}\frac{1}{\rho^n}\sup_{\xi\in\partial D_\rho(0)} p_\alpha(f^{m+1}(\xi))$, for all $0 < \rho < r$, for all α. □

Lemma 1. *Let be A a united locally m-convex algebra and $x \in A$, such that $v(A, \{p_\alpha\}; x) < 1$. The following assertions hold:*
1) There exists $(1-\lambda x)^{-1}$, for all $\lambda \in \overline{D_1(0)}$; $(i.e. \overline{D_1(0)} \subset G_x)$
2) $p_\alpha((1-\lambda x)^{-1}) \le \frac{1}{1 - v(A, \{p_\alpha\}; x)}$, for all α, for all $\lambda \in \overline{D_1(0)}$

Lemma 2. *If A is an unitary locally m-convex algebra and $x \in A$ then there exists $\lim_{n\to\infty}(p_\alpha(x^n))^{\frac{1}{n}}$ and $\lim_{n\to\infty}(p_\alpha(x^n))^{\frac{1}{n}} = \inf_n (p_\alpha(x^n))^{\frac{1}{n}}$, for all α.*
We denote $\inf_n (p_\alpha(x^n))^{\frac{1}{n}} =: l$.

Proof. For any $\epsilon > 0$, there exists $m \in \mathbf{N}$ such that $(p_\alpha(x^m))^{\frac{1}{m}} \le l + \epsilon$, for all α.
For all $n \ge m$, we have $n = mq + r$.
$(p_\alpha(x^n))^{\frac{1}{n}} = (p_\alpha(x^{mq+r}))^{\frac{1}{mq+r}} \le$
$\le (p_\alpha(x^{mq}))^{\frac{1}{mq+r}} (p_\alpha(x))^{\frac{r}{n}} \le$
$\le ((p_\alpha(x^m))^{\frac{1}{m}})^{\frac{n-r}{n}} (p_\alpha(x))^{\frac{r}{n}} \le$
$\le (l+\epsilon)^{1-\frac{r}{n}} (p_\alpha(x))^{\frac{r}{n}}$
Putting $n \to \infty$ it follows that $(p_\alpha(x^n))^{\frac{1}{n}} \le l + \epsilon$.
If $\epsilon \to 0$ we have $(p_\alpha(x^n))^{\frac{1}{n}} \le l$.
It follows that $\overline{\lim} p_\alpha(x^n))^{\frac{1}{n}} = \underline{\lim} p_\alpha(x^n))^{\frac{1}{n}} = l$.
Hence $\lim_{n\to\infty} p_\alpha(x^n))^{\frac{1}{n}} = \inf_n p_\alpha(x^n))^{\frac{1}{n}}$. □

Theorem 3. *Let be A an unitary, locally m-convex and complete algebra. Given $(A, \{p_\alpha\})$, for any $x \in A$ we have that:*

$$\sigma(A; x) \subseteq V(A, \{p_\alpha\}; x).$$

Proof. We know that $D(A, \{p_\alpha\}; 1) = \bigcup_\alpha D(A, p_\alpha; 1)$.
It follows that

$$D(A, \{p_\alpha\}; 1) = \{f \in A' | f(1) = 1,$$

exists α such that $|f(x)| \le p_\alpha(x)$, for all $x \in A\}$. The set $D(A, \{p_\alpha\}; 1)$ is call the set of states of A.
Let be M a subset of A.
We denote by

$$D^l_M(A, \{p_\alpha\}; 1) : = \{f \in D(A, \{p_\alpha\}; 1) | f(yx) = f(y)f(x), \text{ for all } y \in A, x \in M\},$$

the set of multiplicatively to left states with respect to M.
We denote by

$$D^r_M(A, \{p_\alpha\}; 1) : = \{f \in D(A, \{p_\alpha\}; 1) | f(yx) = f(y)f(x), \text{ for all } y \in M, x \in A\},$$

the set of multiplicatively to right states with respect to M.
Let be $D^l_{\{x\}}(A, \{p_\alpha\}; 1)$ (respectively $D^r_{\{x\}}(A, \{p_\alpha\}; 1)$) the set of relatively multiplicatively to left states with respect to x (respectively the set of relatively multiplicatively to right states with respect to x).
Let be $D_{\{x\}}(A, \{p_\alpha\}; 1)$ the set of multiplicatively states with respect to x.
We have $D_{\{x\}}(A, \{p_\alpha\}; 1) = D^l_{\{x\}}(A, \{p_\alpha\}; 1) \cup D^r_{\{x\}}(A, \{p_\alpha\}; 1)$.
We denote $\tilde{\sigma_l}(A; a)$ (respectively $\tilde{\sigma_r}(A; a)$) the anulate spectrum to left of x (respectively anulate spectrum to right

side of x).
$\tilde{\sigma_l}(A;a) = \{\lambda \in \mathbf{C} | \exists\alpha$ s.t. for all $y \in J_l(\lambda l - x), p_\alpha(1-y) \geq 1\}$,
$\tilde{\sigma_r}(A;a) = \{\lambda \in \mathbf{C} | \exists\alpha$ s.t for all $y \in J_r(\lambda l - x), p_\alpha(1-y) \geq 1\}$,
where $J_l(\lambda \cdot 1 - x)$ (respectively $J_r(\lambda \cdot 1 - x)$) represents the left ideal (respectively the right ideal) generated of $\lambda \cdot 1 - x$.
We denote $\tilde{\sigma}(A;x) = \tilde{\sigma_l}(A;x) \cup \tilde{\sigma_r}(A;x)$.
We show that $\tilde{\sigma_l}(A;x) \subseteq \sigma_l(A;x)$, where:

$$\sigma_l(A;x) := \{\lambda | \lambda \cdot 1 - x \text{it is non-invertible to left}\}.$$

Let $\lambda \in \tilde{\sigma_l}(A;x)$. We suppose that $\lambda \notin \sigma_l(A;x)$.
It follows that $t \in A$ such that $t(\lambda \cdot 1 - x) = 1$.
Since $1 \in J_l(\lambda \cdot 1 - x)$, it follows that $p_\alpha(1-1) \geq 1$. Contradiction.
Therefore $\tilde{\sigma_l}(A;x) \subseteq \sigma_l(A;x)$.
We show $C\tilde{\sigma_l}(A;x) \subseteq C\sigma_l(A;x)$.
Let $\lambda \in C\tilde{\sigma_l}(A;x)$. It follows that
for all α, for all $y = z(\lambda \cdot 1 - x), z \in A, p_\alpha(1 - z(\lambda \cdot 1 - x)) < 1$.
We have that $z(\lambda \cdot 1 - x) \in G(A)$. Hence $\lambda \cdot 1 - x \in G^l(A)$.
It follows that $\lambda \notin \sigma_l(A;x)$. Hence $\sigma_l(A;x) \subseteq \tilde{\sigma_l}(A;x)$.
Therefore $\tilde{\sigma_l}(A;x) = \sigma_l(A;x)$.
Analogously we show that $\tilde{\sigma_r}(A;x) = \sigma_r(A;x)$,
where $\sigma_r(A;x) = \{\lambda | \lambda \cdot 1 - x$ it is non-invertible to right$\}$.
We have that $\sigma(A;x) = \sigma_l(A;x) \cup \sigma_r(A;x)$.
It follows that $\sigma(A;x) = \tilde{\sigma_l}(A;x) \cup \tilde{\sigma_r}(A;x)$. (1)
We show $\tilde{\sigma_l}(A;x) = \{f(x) | f \in D^l_{\{x\}}(A, \{p_\alpha\}; 1)\}$.
Let $\lambda \in \tilde{\sigma_l}(A;x)$. It follows that there exists α such that:
$p_\alpha(1-y) \geq 1$, for all $y \in J_l(\lambda \cdot 1 - x)$.
Hence there exists $f \in A'$ such that $f(1) = 1, |f(x)| \leq p_\alpha(x)$, for all $x \in A$, $f|_{J_l(\lambda \cdot 1 - x)} = 0$.
It follows that $f(\lambda \cdot 1 - x) = 0$. Hence $\lambda = f(x)$.
For all $y \in A$, $f(y(\lambda \cdot 1 - x)) = \lambda f(y) - f(yx) = 0$.
It follows that $f(yx) = f(x)f(y) = f(y)f(x)$.
Hence we have $\tilde{\sigma_l}(A;x) \subseteq \{f(x) | f \in D^l_{\{x\}}(A, \{p_\alpha\}; 1)\}$.
We show conversely inclusion. There exists α such that:
$f(1) = 1, |f(x)| \leq p_\alpha(x)$, for all $x \in A$ and $f(yx) = f(x)f(y)$, for all $y \in A$.
Let $\lambda = f(x)$. We have that:
$|f(1 - y(\lambda \cdot 1 - x))| \leq p_\alpha(1 - y(\lambda \cdot 1 - x))$,
$|1 - f(y)f(x) + f(y)f(x)| \leq p_\alpha(1 - y(\lambda \cdot 1 - x))$.
It follows that $p_\alpha(1 - y(\lambda \cdot 1 - x)) \leq 1$.
Hence $\lambda \in \tilde{\sigma_l}(A;x)$. Therefore:
$\{f(x) | f \in D^l_{\{x\}}(A, \{p_\alpha\}; 1)\} \subseteq \tilde{\sigma_l}(A;x)$.
From above:
$\tilde{\sigma_l}(A;x) = \{f(x) | f \in D^l_{\{x\}}(A, \{p_\alpha\}; 1)\}$.
Analogue we show that:
$\tilde{\sigma_r}(A;x) = \{f(x) | f \in D^r_{\{x\}}(A, \{p_\alpha\}; 1)\}$.
It follows from (1) and above that:
$\sigma(A;x) = \{f(x) | f \in D_{\{x\}}(A, \{p_\alpha\}; 1)\}$.
Therefore $\sigma(A;x) \subseteq V(A, \{p_\alpha\}; 1)$. □

Theorem 4. *Let be A a locally m-convex unitary complete algebra. Then:*

$$\rho(x) = \sup_\alpha \lim_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}}$$

Proof. We suppose that $\rho(x)$ is bounded. Let $r > \rho(x), \lambda \in \mathbf{C}$
such that $|\lambda| > \rho(x)$ (hence $|\lambda| > 0$) and $\lambda \notin \sigma(A;x)$,
hence $\lambda \cdot 1 - x \in G(A)$ It follows that $\lambda(1 - \frac{1}{\lambda}x) \in G(A)$.
It follows that $1 - \frac{1}{\lambda}x \in G(A)$, for all $|\frac{1}{\lambda}| \leq \frac{1}{r}$, therefore $\overline{D_{\frac{1}{r}}(0)} \subseteq G_x$.
It follows that from corollary when $m = 0$:

$$P_\alpha(x^n) \leq \frac{1}{\rho^n} \sup_{\xi \in \partial D_\rho(0)} p_\alpha((1 - \xi x)^{-1}), \forall \alpha, \forall 0 < \rho < \frac{1}{r}.$$

Hence $(p_\alpha(x^n))^{\frac{1}{n}} \leq \frac{1}{\rho} \sup_{\xi \in \partial D_\rho(0)} p_\alpha((1 - \xi x)^{-1})$, for all $0 < \rho < \frac{1}{r}$, for all α.
It follows that $\lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}} \leq \dfrac{1}{\rho}$. Putting $\rho \nearrow \frac{1}{r}$ we have $\sup\limits_\alpha \lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}} \leq \rho(x)$. $(*)$
We show above theorem that:

$$\sigma(A;x) = \{f(x) | f \in D_{\{x\}}(A, \{p_\alpha\}, 1)\}.$$

Let $\lambda \in \sigma(A;x)$. It follows that there exists $f \in D_{\{x\}}(A, \{p_\alpha\}, 1)$ such that $\lambda = f(x)$.
Hence $\lambda^n = (f(x))^n = f(x^n)$,
because f is relatively multiplicatively with respect to x.
it follows that there exist α such that:

$$|\lambda^n| = |f(x^n)| \leq (p_\alpha(x^n)), \quad \text{for all } n \in \mathbf{N}.$$

Hence there exists α such that $|\lambda| \leq \lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}}$.
Therefore $|\lambda| \leq \sup\limits_\alpha \lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}}$.
Hence $\rho(x) \leq \sup\limits_\alpha \lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}}$. $(**)$
From $(*)$ and $(**)$ we have that $\rho(x) = \sup\limits_\alpha \lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}}$.
Therefore $\rho(x) = \infty$ it follows from $(**)$ that $\lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}} = \infty$.
Therefore $\rho(x) = \sup\limits_\alpha \lim\limits_{n\to\infty} (p_\alpha(x^n))^{\frac{1}{n}}$. □

Theorem 5. *Let be A an unitary, locally m-convex, complete algebra. For all $a \in A$ we have:*

$$co\,\sigma(A;a) \subseteq \bigcap \{V(A, \{p_\alpha\}; a) | \{p_\alpha\} \in P(A)\} \subseteq \overline{co}\,\sigma(A;a).$$

Proof. From Theorem 3 we have that:

$$co\,\sigma(A;a) \subseteq \bigcap \{V(A, \{p_\alpha\}; a) | \{p_\alpha\} \in P(A)\}.$$

If $\overline{co}\ \sigma(A;a)$ it is not complex plan, then for all $\lambda \notin \overline{co}\,\sigma(A;a)$ there exists an open disk D_λ, centered in λ, such that D_λ can be separated strictly from $\overline{co}\ \sigma(A;a)$ through straight line L and $\sigma(A;a) = \bigcup\limits_{\alpha} \sigma(\overline{A_\alpha};a_\alpha)$.

D_λ is separated strictly from $\sigma(\overline{A_\alpha};a_\alpha)$, for all α, through straight line L. Since for all α we have $\sigma(\overline{A_\alpha};a_\alpha)$ is a compact set, hence there exists $D_\alpha \supseteq \sigma(\overline{A_\alpha};a_\alpha)$, which is strictly separate from D_λ through line L.
We have from [2] that for all α, there exists a norm $\|\cdot\|'_\alpha$ equivalent with $\|\cdot\|_\alpha$ on $\overline{A_\alpha}$ such that:

$$\sigma(\overline{A};a_\alpha) \subseteq V(A,\|\cdot\|'_\alpha;a_\alpha) \subseteq D_\alpha.$$

We define seminorm p'_α on A through $p'_\alpha = \|x_\alpha\|'_\alpha$.
Since $V(\overline{A},\|\cdot\|'_\alpha;a_\alpha) = V(A,\|\cdot\|'_\alpha;a_\alpha)$, it is clear that the family $\{p'_\alpha\} \in P(A)$ and

$$V(A,\{p'_\alpha\};a) = \bigcup_{\alpha}\{V(A,\|\cdot\|'_\alpha;a_\alpha)\}.$$

Hence D_λ it is strictly separated by $\bigcap\limits_{\alpha}\{V(A,\{p_\alpha\};a)|\{p_\alpha\} \in P(A)\}$, and therefore $\bigcap\limits_{\alpha}\{V(A,\{p_\alpha\};a)|\{p_\alpha\} \in P(A)\} \subseteq \overline{co}\,\sigma(A;a)$. □

References

[1] J.R. Giles & D.O. Koehler, "On Numerical Ranges of locally m-convex Algebras", Pacific Journal of Mathematics, 1973.

[2] F.F. Bonsal & J.Duncan, "Numerical Range of Operators on Normed Spaces and of Elements of Normed Algebras", London Math. Soc., Cambridge, 1971.

[3] H.F. Bohnenblust & S.Karlin, "Geometrical properties of the unit sphere of Banach algebras', Ann. of Math, 1955.

[4] C. Apostol, "C^*-algebras and their representation", J. London Math. Soc., 1971.

[5] E.A. Michael, "Locally multiplicativelyly convex topological algebras",Amer. Math. Soc., 1952.

[6] B. Sims, "A characterization of Banach-star-algebras by numerical range", Bull. Austral. Math., 1971.

[7] C.A. Berger & J.G. Stampfli, "Mapping theoremes for the numerical range", Amer. L. Math., 1967.

[8] E. Berkson, "Some characterizations of C^*-algebras", Ill. J. Math., 1966.

[9] D.M. Petrovai, "Imaginea numerică în algebre local multiplicativ convexe," Editura Matrixrom, 2007.