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Numerical solutions of first order iterative functional-differential equations by spline functions of even degree

Nicola Oprea
"Petru Maior" University of Tîrgu Mureş, Romania
Nicolae Iorga street nr. 1, 540088
noprea@engineering.upm.ro

Abstract

This paper presents a numerical method for the approximate solution of a first order iterative functional-differential equations. This method is essentially based on the the natural spline functions of even degree introduced by using the derivative- interpolating conditions on simple knots.

1 Introduction

Spline theory is nowadays a very active field of approximation theory and many desirable advantages exist for differential equations problems. Since they are easy to evaluate and manipulate on computer a lot of applications in the numerical solution of a variety of problem in applied mathematics have been found.

The spline functions of even degree are defined in a similar manner with that for odd degree spline functions, but using the derivative-interpolating conditions. These spline functions preserve all the remarkable extremal and convergence properties of the odd degree splines and are very suitable for the numerical solutions of the differential equation problems.

In this paper we consider a spline approximation method for the numerical solutions of first order iterative functional-differential equations. For details of the theory of spline functions of even degree we refer to the monographs [6], [10] and for an exhaustive literature on spline functions and their applications we refer to [1], [2], [3]. The purpose of the present study is to extend the results of [5], [8], [9] for the first order iterative functional-differential equations. We shall develop some theory, algorithms and a very efficient procedure to use this spline functions of even degree for the numerical solutions of this class of differential equations.

2 Basic definitions and properties of even degree splines

Let Δ_n be the following partition of the real axis

$$\Delta_n : -\infty = t_0 < a = t_1 < ... < t_n = b < t_{n+1} = +\infty$$

and let m, n be two given natural numbers, satisfying the conditions $n \geq 1,\ m \leq n+1$. One denotes by I_k the following subintervals

$$I_k := [t_k, t_{k+1}[,\ k = \overline{1,n},\ I_0 :=]t_0, t_1[.$$

Following [2] we present some definitions and theorems.

Definition 1. *For the couple (m, Δ_n) a function $s : \mathbb{R} \to \mathbb{R}$ is called a natural spline function of even degree $2m$ if the following conditions are satisfied:*
1^0 $s \in C^{2m-1}(\mathbb{R})$,
2^0 $s\mid_{I_k} \in \mathcal{P}_{2m},\ k = \overline{1,n}$,
3^0 $s\mid_{I_0} \in \mathcal{P}_m,\ s\mid_{I_n} \in \mathcal{P}_m$,
where $\mathcal{P}_k$ represents the set of algebraic polynomials of degree $\leq k$.

We denote by $\mathcal{S}_{2m}(\Delta_n)$ the linear space of natural polynomial splines of even degree $2m$ with the simple knots $t_1, ..., t_n$.
We now show that $\mathcal{S}_{2m}(\Delta_m)$ is a finite dimensional linear space of functions and we give a basis of it.

Theorem 1. *Any element $s \in \mathcal{S}_{2m}(\Delta_n)$ has the following representation*

$$s(t) = \sum_{i=0}^{m} A_i t^i + \sum_{k=1}^{n} a_k (t - t_k)_+^{2m},$$

where the real coefficients $(A_i)_0^m$ are arbitrary, and the coefficients $(a_k)_1^n$ satisfy the conditions

$$\sum_{k=1}^{n} a_k t_k^i = 0,\ i = \overline{0, m-1}.$$

Keywords: *first order iterative functional-differential equations, natural spline functions*

Remark 1. *If $n+1=m$, then $a_k=0$, $k=\overline{1,n}$.*

Theorem 2. *Suppose that $n+1 \geq m$, and let $f:[t_1,t_n] \to \mathbb{R}$ be a given function such that $f'(t_k)=y'_k,\ k=\overline{1,n}$, and $f(t_1)=y_1$, where $y'_k,\ k=\overline{1,n}$, and y_1 are given real numbers. Then there exists a unique spline function $s_f \in \mathcal{S}_{2m}(\Delta_n)$, such that the following derivative-interpolating conditions*

$$s_f(t_1)=y_1, \tag{1}$$

$$s'_f(t_k)=y'_k,\ k=\overline{1,n}, \tag{2}$$

hold.

Corollary 1. *There exists a unique set of $n+1$ fundamental natural polynomial spline functions $S_k \in \mathcal{S}_{2m}(\Delta_n),\ k=\overline{1,n}$, and $s_0 \in \mathcal{S}_{2m}(\Delta_n)$ satisfying the conditions:*

$$\begin{array}{lll} s_0(t_1)=1, & s'_0(t_k)=0, & k=\overline{1,n}, \\ S_k(t_1)=0, & S'_k(t_i)=\delta_{ik}, & i,k=\overline{1,n}. \end{array}$$

It is clear that the functions $\{s_0, S_k,\ k=\overline{1,n}\}$, form a basis of the linear space $\mathcal{S}_{2m}(\Delta_n)$, and for s_f we obtain the representation

$$s_f(t)=s_0(t)f(t_1)+\sum_{k=1}^{n} S_k(t)f'(t_k).$$

But because $s_0(t)=1$, it follows that

$$s_f(t)=f(t_1)+\sum_{k=1}^{n} S_k(t)f'(t_k).$$

Let us introduce the following sets of functions

$W_2^{m+1}(\Delta_n) := \{g:[a,b]\to\mathbb{R} \mid g^{(m)}$
abs.cont.on I_k and $g^{(m+1)} \in L_2[a,b]\}$,
$W_2^{m+1}[a,b] := \{g:[a,b]\to\mathbb{R} \mid g^{(m)}$
abs.cont.on$[a,b]$ and $g^{(m+1)} \in L_2[a,b]\}$,
$W^{m+1}_{\overset{0}{2,f}}(\Delta_n) := \{g \in W_2^{m+1}(\Delta_n) \mid g'(t_k)=f'(t_k)\}$,
$W^{m+1}_{2,f}(\Delta_n) := \{g \in W_2^{m+1}(\Delta_n) \mid g(t_0)=f(t_0)\}$.

Theorem 3. *(Minimal norm property). If $s \in \mathcal{S}_{2m}(\Delta_n) \cap W^{m+1}_{\overset{0}{2,f}}(\Delta_n)$, then*

$$\left\|s^{(m+1)}\right\|_2 \leq \left\|g^{(m+1)}\right\|_2,\ \forall g \in W^{m+1}_{\overset{0}{2,f}}(\Delta_n),$$

holds, $\|\cdot\|_2$ being the usual L_2-norm.

For any function $f \in W_2^{m+1}(\Delta_n)$, we have the following corollaries.

Corollary 2.

$$\left\|f^{(m+1)}\right\|_2^2 = \left\|s_f^{(m+1)}\right\|_2^2 + \left\|f^{(m+1)}-s_f^{(m+1)}\right\|_2^2.$$

Corollary 3. $\left\|s_f^{(m+1)}\right\|_2 \leq \left\|f^{(m+1)}\right\|_2.$

Corollary 4. $\left\|f^{(m+1)}-s_f^{(m+1)}\right\|_2 \leq \left\|f^{(m+1)}\right\|_2.$

Remark 2. *If $\widetilde{s} := s_f+p_m$, where $p_m \in \mathcal{P}_m$, it follows $\left\|\widetilde{s}^{(m+1)}\right\|_2 \leq \left\|f^{(m+1)}\right\|_2.$*

Theorem 4. *(Best approximation property). If $f \in W_2^{m+1}(\Delta_n)$ and $s_f \in \mathcal{S}_{2m}(\Delta_n)$ is the derivative-interpolating spline function of even degree, then, for any $s \in \mathcal{S}_{2m}(\Delta_n)$ the relation*

$$\left\|s_f^{(m+1)}-f^{(m+1)}\right\|_2 \leq \left\|s^{(m+1)}-f^{(m+1)}\right\|_2$$

holds.

Remark 3. *If $s_f - s \in \mathcal{P}_m$ then*

$$\left\|s_f^{(m+1)}-f^{(m+1)}\right\|_2 = \left\|s^{(m+1)}-f^{(m+1)}\right\|_2.$$

3 The numerical solutions of first order iterative functional-differential equations by spline functions of even degree

Let us consider the following iterative functional-differential problem

$$y'(t)=f(t,y(t),y(y(t))),\ a \leq t \leq b, \tag{3}$$

$$y(t)=\varphi(t),\ a_1 \leq t \leq a, \tag{4}$$

with the following assumptions:

(C_1) $a,b,a_1 \in \mathbb{R},\ a_1 \leq a < b$;

(C_2) $f \in C([a,b]\times[a_1,b]^2,\mathbb{R})$;

(C_3) $\varphi \in C([a_1,a],[a_1,b])$.

By a solution of the problem (3)–(4) we understand a function $y \in C([a_1,b],[a_1,b]) \cap C^1([a,b],[a_1,b])$ which satisfies (3)–(4).

We suppose that $f:[a,b]\times[a_1,b]^2 \to \mathbb{R}$ satisfies all the conditions assuring the existence and uniqueness of the solution y of the problem (3)–(4).

We propose an algorithm to approximate the solution y of the problem (3)–(4) by spline functions of even degree $s \in \mathcal{S}_{2m}(\Delta_n)$, where Δ_n is a partition of $[a,b]$ and m,n are two integers satisfying the conditions $n \geq 1$ and $m \leq n+1$.

Theorem 5. *If y is the exact solution of the problem (3)–(4), then, there exists a unique spline function $s_y \in \mathcal{S}_{2m}(\Delta_n)$ such that:*

$$\begin{array}{l} s_y(t_1)=y(t_1)=\varphi(t_1), \\ s'_y(t_k)=y'(t_k),\ k=\overline{1,n}, \end{array} \tag{5}$$

The assertion of this theorem is a direct consequence of Theorem 2 by substituting t_1 by a and f by y.

Denoting $y_k := y(t_k)$ and $\overline{y}_k = y(y(t_k)),\ k = \overline{1,n}$, we have

$$s_y(t_1) = y_1$$
$$s'_y(t_k) = f(t_k, y_k, \overline{y}_k),\ k = \overline{1,n}.$$

Corollary 5. *If the functions $\{s_0, S_k, k = \overline{1,n}\}$ are the fundamental spline functions in $\mathcal{S}_{2m}(\Delta_n)$, then we can write*

$$s_y(t) = \varphi(a) + \sum_{k=1}^{n} S_k(t) f(t_k, y_k, \overline{y}_k),\ y \in [a,b]. \quad (6)$$

The unknown values $y_k, \overline{y}_k,\ k = \overline{1,n}$ are to be determined as we shall show later. Before giving an algorithm to determine these values, we shall give the following estimation error and convergence theorem (see [2]).

Theorem 6. *If $y \in W_2^{m+1}[a,b]$ is the exact solutions of the problem (3)–(4) and s_y is the spline approximating solution for y, the following estimations hold:*

$$\left\| y^{(k)} - s_y^{(k)} \right\|_\infty \le \sqrt{m}(m-1)...k\Delta_n^{m-k+\frac{1}{2}} \left\| y^{(m+1)} \right\|_2,$$

for $k = 1, 2, ..., m$, where $\|\Delta_n\| := \max\limits_{i=\overline{2,n}} \{t_i - t_{i-1}\}$.

Corollary 6. *If $y \in W_2^{m+1}[a,b]$, we have*

$$\|y - s_y\|_\infty \le (b-a)\sqrt{m}(m-1)! \left\| y^{(m+1)} \right\|_2 \|\Delta_n\|^{m-\frac{1}{2}}.$$

Corollary 7. $\lim\limits_{\|\Delta_n\| \to 0} \left\| y^{(k)} - s_y^{(k)} \right\|_\infty = 0,\ k = \overline{1,m}.$

4 Effective development of the algorithm

For any $t \in [a,b]$, we suppose that $y(t) \approx s_y(t)$.

If we denote, as usual, $e(t) := y(t) - s_y(t),\ t \in [a,b]$,we have

$$|e(t)| \le \sqrt{m}(m-1)! \|\Delta_n\|^{m-\frac{1}{2}} \left\| y^{(m+1)} \right\|_2,$$

or

$$|e(t)| = O(\|\Delta_n\|^{m-\frac{1}{2}}),\ \forall t \in [a,b].$$

If we denote

$$w_i := s_y(t_i), \quad e_i := e(t_i) = y(t_i) - s_y(t_i), i = \overline{1,n},$$
$$\overline{w}_i := y(s_y(t_i)), \quad \overline{e}_i := \overline{e}(t_i) = y(y(t_i)) - y(s_y(t_i)),$$

$i = \overline{1,n}$, then we have $y_i = w_i + e_i$, $\overline{y}_i = \overline{w}_i + \overline{e}_i$ where

$$w_i = y_1 + \sum_{k=1}^{n} S_k(t_i) f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k),\ i = \overline{1,n}, \quad (7)$$

$$\overline{w}_i = y\left(y_1 + \sum_{k=1}^{n} S_k(t_i) f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k) \right),$$

$i = \overline{1,n}$. In what follows, we suppose that in (3)–(4) the functions $f : D \subset \mathbb{R}^4 \to \mathbb{R}$ $(D \subset [a,b] \times \mathbb{R}^3)$,

$$\frac{\partial f(t,u,v)}{\partial u}, \ \frac{\partial f(t,u,v)}{\partial v}$$

are continuous. Thus,

$$\begin{aligned} f(t_k, y_k, \overline{y}_k) &= f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k) \\ &= f(t_k, w_k, \overline{w}_k) + e_k \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial u} + \\ &+ \overline{e}_k \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial v} \end{aligned}$$

where

$$\min(w_k, w_k + e_k) < \xi_k < \max(w_k, w_k + e_k),$$
$$\min(\overline{w}_k, \overline{w}_k + \overline{e}_k) < \eta_k < \max(\overline{w}_k, \overline{w}_k + \overline{e}_k).$$

We can write the system (7) in the form

$$w_i = y_1 + \sum_{k=1}^{n} S_k(t_i) f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k) + E_i,$$
$$i = \overline{1,n},$$
$$\overline{w}_i = y\left(y_1 + \sum_{k=1}^{n} S_k(t_i) f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k) \right) + \overline{E}_i,$$
$$i = \overline{1,n},$$

where

$$\begin{aligned} E_i &= \sum_{k=1}^{n} S_k(t_i) e_k \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial u} + \\ &+ \sum_{k=1}^{n} S_k(t_i) \overline{e}_k \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial v} = O(\|\Delta_n\|^{m-\frac{1}{2}}), \\ \overline{E}_i &= y\left(\sum_{k=1}^{n} S_k(t_i) e_k \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial u} + \right. \\ &\left. + \sum_{k=1}^{n} S_k(t_i) \overline{e}_k \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial v} \right) = O(\|\Delta_n\|^{m-\frac{1}{2}}), \end{aligned}$$

supposing that

$$\left| \frac{\partial f(t,u,v)}{\partial u} \right| \le M, \left| \frac{\partial f(t,u,v)}{\partial v} \right| \le N \quad (8)$$

on D. Obviously, $E_i \to 0$ and $\overline{E}_i \to 0$ for $\|\Delta_n\| \to 0$.

Now, we have to solve the following nonlinear system:

$$\begin{cases} w_i = y_1 + \sum\limits_{k=1}^{n} S_k(t_i) f(t_k, w_k, \overline{w}_k),\ i = \overline{1,n}, \\ \overline{w}_i = y\left(y_1 + \sum\limits_{k=1}^{n} S_k(t_i) f(t_k, w_k, \overline{w}_k) \right),\ i = \overline{1,n}\ . \end{cases} \quad (9)$$

Let us denote:

$$w := (w_1, ..., w_n),\ \overline{w} := (\overline{w}_1, ..., \overline{w}_n),\ W = (w, \overline{w}),$$

$$H_i(w, \overline{w}) := y_1 + \sum_{k=1}^{n} S_k(t_i) f(t_k, w_k, \overline{w}_k),\ i = \overline{1, n},$$

$$\overline{H}_i(w, \overline{w}) := y\left(y_1 + \sum_{k=1}^{n} S_k(t_i) f(t_k, w_k, \overline{w}_k)\right)$$

$$H(W) := H(w, \overline{w})$$
$$:= (H_1(w, \overline{w}), ..., H_n(w, \overline{w}), \overline{H}_1(w, \overline{w}), ..., \overline{H}_n(w, \overline{w}))$$

and

$$A = \begin{pmatrix} \frac{\partial H_1(w,\overline{w})}{\partial w_1} & \dots & \frac{\partial H_1(w,\overline{w})}{\partial w_n} & \dots & \frac{\partial H_1(w,\overline{w})}{\partial \overline{w}_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial H_n(w,\overline{w})}{\partial w_1} & \dots & \frac{\partial H_n(w,\overline{w})}{\partial w_n} & \dots & \frac{\partial H_n(w,\overline{w})}{\partial \overline{w}_n} \\ \frac{\partial \overline{H}_1(w,\overline{w})}{\partial w_1} & \dots & \frac{\partial \overline{H}_1(w,\overline{w})}{\partial w_n} & \dots & \frac{\partial \overline{H}_1(w,\overline{w})}{\partial \overline{w}_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial \overline{H}_n(w,\overline{w})}{\partial w_1} & \dots & \frac{\partial \overline{H}_n(w,\overline{w})}{\partial w_n} & \dots & \frac{\partial \overline{H}_n(w,\overline{w})}{\partial \overline{w}_n} \end{pmatrix}$$

Shortly, we write the system (9) by

$$W = H(W). \tag{10}$$

In order to investigate the solvability of the nonlinear system (10) we shall use a classical theorem.

Theorem 7. *Let $\Omega \subset \mathbb{R}^{2n+2}$ be a bounded domain and let $H : \Omega \to \Omega$ be a vector function defined by*

$$W = (w, \overline{w}) \quad \longmapsto \quad (H_1(w, \overline{w}), ..., H_n(w, \overline{w}), \overline{H}_1(w, \overline{w}), ..., \overline{H}_n(w, \overline{w})) = H(W).$$

If the functions H, and $\dfrac{\partial H}{\partial W}$, are continuous in Ω, then there exists in Ω a fixed point W^* of H, i.e. $W^* = H(W^*)$, which can be found by iterations. $W^* = \lim_{n\to\infty} W^{(n)}$, $W^{(k)} := H(W^{(k-1)})$, $k = 1, 2, ...$, $W^{(0)} \in \Omega$ (arbitrary). If in addition $\|A\| \le L < 1$, for any iteration $W^{(k)}$, the following estimation holds:

$$\left\| W - W^{(k)} \right\| \le \frac{L^k}{1 - L} \left\| W^{(1)} - W^{(0)} \right\|.$$

Taking in consideration the expression of H, the matrix A is $A = SF$, where

$$S = \begin{pmatrix} S_1(t_1) & \cdots & S_n(t_1) & \cdots & S_n(t_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_1(t_n) & \cdots & S_n(t_n) & \cdots & S_n(t_n) \\ y(S_1(t_1)) & \cdots & y(S_n(t_1)) & \cdots & y(S_n(t_1)) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y(S_1(t_n)) & \cdots & y(S_n(t_n)) & \cdots & y(S_n(t_n)) \end{pmatrix}$$

and F is the diagonal matrix with the following elements:

$$\frac{\partial f(t_k, w_k, \overline{w}_k)}{\partial w_k},\ \frac{\partial f(t_k, w_k, \overline{w}_k)}{\partial \overline{w}_k}, k = \overline{1, n}.$$

Theorem 8. *Suppose that there exists the constants M, N such that (8) holds and*

$$|f(t, u, v)| \le P,\ \forall (t, u, v) \in D.$$

If $M \le \|S\|^{-1}$, then the system (9) has a solution which can be found by iterations.

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