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SOME CONSIDERATIONS ON PARAMETRIC MINIMUM PROBLEMS

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ABSTRACT

A sequence of parametric minimum problems related to a "limit" minimum problem is considered. A comparison on sufficient conditions given in [1] *and* [3] *is studied.*

Keywords: minimum problems, $\Gamma-$convergence

1 Introduction

Let X be a Hausdorff topological space and let us consider, for any $n \in \mathbb{N}$, the following minimum problems:

$(M)_n$ Find an element $a_n \in X$ such that

$$f_n(a_n) \le f_n(b),\ \forall b \in X,$$

(M) Find an element $a \in X$ such that

$$f(a) \le f(b),\ \forall b \in X,$$

where $f_n : X \to \mathbb{R}$, $f : X \to \mathbb{R}$.

The aim is to compare conditions on the "convergence" of $(f_n)_n$ to f in order to obtain a convergence result for the solutions of $(M)_n$ to solutions of (M).

We shall translate conditions given in [1] and [3] to our specific problems. Generally, equilibrium problems deal with bifunctions $g : X \times X \to \mathbb{R}$, therefore we consider in particular $g(a, b) = f(b) - f(a)$.

2 Main part

Proposition 1. *Let $(a_n)_n$ be a sequence of solutions of $(M)_n$ and let $a_n \to a$. Suppose that f is lower semicontinuous at a and f_n, f verify condition $(\mathbf{C})_{\mathbf{inf}}$*

$$\liminf_n [f_n(b) - f(b) - f_n(a_n) + f(a_n)] \le 0, \forall b \in X.$$

Then, limit a is a solution of (M).

Proof. Let $b \in X$ be arbitrary. We have

$$\begin{aligned} 0 &\le \liminf_n [f_n(b) - f_n(a_n)] \\ &\le \limsup_n [f(b) - f(a_n)] \\ &\quad + \liminf_n \big(f_n(b) - f_n(a_n) - f(b) + f(a_n)\big) \\ &\le \limsup_n [f(b) - f(a_n)] \\ &= f(b) - \liminf_n f(a_n) \le f(b) - f(a). \end{aligned}$$

□

Proposition 2. *Let $(a_n)_n$ be a sequence of solutions of $(M)_n$ and let $a_n \to a$. Suppose that f_n, f verify condition $(\mathbf{C})_{\mathbf{sup}}$*

$$\limsup_n [f_n(b) - f(b) - f_n(a_n) + f(a)] \le 0, \forall b \in X.$$

Then, limit a is a solution of (M).

Proof. See [3] for the general case.

$$\begin{aligned} 0 &\le \limsup_n [f_n(b) - f_n(a_n)] \\ &= f(b) - f(a) \\ &\quad + \limsup_n \big(f_n(b) - f_n(a_n) - f(b) + f(a)\big) \\ &\le f(b) - f(a), \forall\, b \in X. \end{aligned}$$

□

It is evident that one can replace $\limsup_n$ with $\liminf_n$ in $(\mathbf{C})_{\mathbf{sup}}$ and the proof does not change. Starting from the trivial situation when $f = 0$, one can remark that $(\mathbf{C})_{\mathbf{inf}}$ is less restrictive than

$(\mathbf{C})_{\mathbf{sup}}$ so Proposition 1 can be applied although $\limsup_n[f_n(b) - f_n(a_n)] > 0$, for some $b \in X$.

Condition $(\mathbf{C})_{\mathbf{sup}}$ implies $(\mathbf{C})_{\mathbf{inf}}$ and lower semicontinuity of f at a, when $(a_n)_n$ are solutions of $(M)_n$ and $a_n \to a$. Chose $b := a_n$ in $(\mathbf{C})_{\mathbf{sup}}$, hence $\limsup_n[-f(a_n) + f(a)] \le 0$, i.e. f is lower semicontinuous at a.

Let us remind the definition of $\Gamma-$convergence (see [2]).

Definition 1. *A sequence $(f_n)_n$, $f_n : X \to \overline{\mathbb{R}}$ is said to Γ-converges in X to $f : X \to \overline{\mathbb{R}}$ if, for all $x \in X$ one has*

1. *for each $(x_n)_n$ convergent to x it results in*
$$f(x) \le \liminf_n f_n(x_n);$$
2. *there exists a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to x such that*
$$f(x) \ge \limsup_n f_n(x_n).$$

Suppose that $(f_n)_{n\in\mathbb{N}}$ $\Gamma-$converges to f. Is condition $(\mathbf{C})_{\mathbf{inf}}$ verified ? Let us try with the following increases:

$$\begin{aligned}
&\liminf_n[f_n(b_n) - f(b) - f_n(a_n) + f(a_n)] \\
&\quad \le \limsup_n[f_n(b_n) - f(b)] \\
&\qquad + \liminf_n[-f_n(a_n) + f(a_n)] \\
&\quad \le -\liminf_n f_n(a_n) + \liminf_n f(a_n) \\
&\quad \le -f(a) + \liminf_n f(a_n).
\end{aligned}$$

The answer is yes if

$$\liminf_n f(a_n) \le f(a). \tag{1}$$

Take, for example, $f_n(x) = -\cos(2\pi n x)$, $x \in [0,1] = X$. In this case $(f_n)_n$ $\Gamma-$converges to the constant function $f = -1$ so (1) applies trivially.

Take $f_n(x) = f_1(nx)$, $x \in \mathbb{R} = X$, where $f_1(x) = \begin{cases} 1, x = 1 \\ -1, x = -1 \\ 0, \text{ otherwise }. \end{cases}$ Then, (f_n) $\Gamma-$converges to f, where $f(x) = \begin{cases} 0, & x \neq 0 \\ -1, & x = 0. \end{cases}$ Remark that (1) does not apply since

$$\liminf_n f(a_n) = 0 > -1 = f(0).$$

Condition $(\mathbf{C})_{\mathbf{inf}}$ is not verified. Indeed, $a_n = -1/n$ are solutions for $(M)_n$, $a = 0$ is the solution for (M) while, for $b = -1$ we have

$$\begin{aligned}
&\liminf_n[f_n(-1) - f(-1) - f_n(-1/n) + f(-1/n)] \\
&\qquad = 0 - 0 - (-1) + 0 > 0.
\end{aligned}$$

In this case, the conclusion in Proposition 1 is obtained by Theorem 1.21 in [2]. It is worth nothing that in Proposition 1 there are no compactness assumptions.

3 Conclusions

The notion of $\Gamma-$convergence is a powerful tool in the study of variational problems. It involves the study of parametric minimum problems. We tried to take a closer look to this type of convergence, known also as epi-convergence, when we particularized the parametric equilibrium problems in [1] and the parametric quasivariational inequalities in [3].

References

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