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Examples of Lebesgue Integrable Functions which is not Riemann Integrable

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Abstract

We present some examples of Lebesgue integrable functions which is not Riemann integrable.

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1 Integral of a numerical function with respect to a positive measure

We will define integral of a measurable function in three successive parts: first for simple functions, next for positive measurable functions and finally for arbitrary measurable functions.

We will must to be show that every new definition is compatible with precedent. We will give first properties of integral $\int_X f d\mu$ for $f \in \mathcal{S}(X, \mathcal{A}, \mu)$ from which we will deduce properties of integral $\int_X f d\mu$ for $f \in \mathcal{M}^+(X, \mathcal{A}, \mu)$, then we deduce properties of integral $\int_X f d\mu$ for $f \in \mathcal{M}(X, \mathcal{A}, \mu)$. The most important results of this section are: sufficient conditions of integrability, algebraical properties of integrable functions, property of monotony of integral, the limit under integral. (Theorem of monotonous convergence, dominated convergence theorem etc.).

Definition 1. Let $(X, \mathcal{A}, \mu)$ be a space with measure, $f : X \to \mathbb{R}$ a simple function and $\sum_{i=1}^{n} c_i \varphi_{A_i}$ a canonical representation of f. If expression $\sum_{i=1}^{n} c_i \mu(A_i)$ has sense is noted by $\int_X f(x) d\mu$, $\int_X f d\mu$ or $\int f d\mu$ and this called integral of f (on X). We will say that f has integral. If f has integral and its integral is finite i.e. $|\int_X f d\mu| < +\infty$, f is called integrable (on X). We will say that f has integral on $A \in \mathcal{A}$ if the function $f \cdot \varphi_A$ has integral; $\int_X f \varphi_A d\mu$ is noted by $\int_A f d\mu$ and is called integral of f on A. If moreover $|\int_A f d\mu| < +\infty$ f is called integrable on A. If $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{L}, \mu)$ integral of f defined above is called Lebesgue integral. Let $(X, \mathcal{A}, \mu)$ be a space with measure. We will denote with $\mathcal{S}(X, \mathcal{A}, \mu)$ the set of all simple numerical functions, $\mathcal{M}(X, \mathcal{A}, \mu)$ the set of all measurable numerical functions and with $\mathcal{M}^+(X, \mathcal{A}, \mu)$ the set of all positive measurable numerical functions.

2 The elementary properties

Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f = \sum_{i=1}^{n} c_i \varphi_{A_i}$ a simple function defined on X. Then the following assertions hold:

1. f integrable if and only if for any $i = \overline{1, n}$ with $\mu(A_i) = +\infty$ we have $c_i = 0$ if and only if for any $i = \overline{1, n}$ with $c_i \neq 0$ we have $\mu(A_i) < +\infty$.

2. If $f \geq 0$ (resp. $f \leq 0$) it follows that $\int_X f d\mu$ has sense and we have $\int_X f d\mu \geq 0$ (resp. ≤ 0), hence f is integrable if and only if $\int_X f d\mu < +\infty$ (resp $> -\infty$).

3. If f is integrable (on X) then f is integrable on any set $A \in \mathcal{A}$.

4. If $A \in \mathcal{A}$ and $\mu(A) < +\infty$, particularly if $\mu(X) < +\infty$ it follows that f is integrable on A.

5. If $A \in \mathcal{A}$ and $\mu(A) = 0$ it follows that f is integrable on A and $\int_A f = 0$.

6. If $A \in \mathcal{A}$ and $f = \varphi_A$ we have $\int_X f d\mu = \mu(A)$, hence f is integrable if and only if $\mu(A) < +\infty$

Examples. 1. Let $f : \mathbb{R} \to \mathbb{R}$ function of Dirichlet, hence $f = 1 \cdot \varphi_Q + 0 \cdot \varphi_{\mathbb{R} \setminus Q} =$ and hence $\int_{\mathbb{R}} f d\lambda = 1 \cdot \mu(Q) + 0 \cdot \mu(\mathbb{R} \setminus Q) = 0$, therefore f is integrable Lebesgue.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function of Heaviside i.e.

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}.$$

Then $f = \varphi_{\mathbb{R}_+} = 1 \cdot \varphi_{\mathbb{R}_+} + 0 \cdot \varphi_{\mathbb{R}^*_-}$ hence $\int_{\mathbb{R}} f d\mu = 1 \cdot \mu(\mathbb{R}_+) + 0 \cdot \mu(\mathbb{R}^*_-) = +\infty$. Therefore f has integral but is not Lebesgue integrable. Is seen that $\int_{[-1,1]} f\mu = 1$, hence f is integrable on $[-1,1]$.

3. Let $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be, where $\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}_+$, $\mu(A) = \text{card}A$.

Then for any $A \subseteq \mathbb{N}$ we have $\int_{\mathbb{N}} 1_A d\mu = \text{card}A$, hence f is integrable if and only if card $(A) < \infty$.

4. Let sign:$\mathbb{R} \to \mathbb{R}$ the signum function, i.e.

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}, \text{ i.e } \text{sign}(x) = -1\varphi_{\mathbb{R}^*_-} + 0 \cdot \varphi_{\{0\}} + 1 \cdot \varphi_{\mathbb{R}^*_+}$$

Then sum $-1 \cdot \lambda(\mathbb{R}^*_-) + 0 \cdot \lambda(\{0\}) + 1 \cdot \lambda(\mathbb{R}^*_+)$ has not sense, hence function sign has not Lebesgue integral.

Proposition 1. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ a simple function which has integral. Then integral of f is a number unique determined by f.

Proposition 2. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ a simple function. Then f is integrable if and only if $|f|$ is integrable.

Proposition 3. Let $(X, \mathcal{A}, \mu)$ be a space with measure, $f, g : X \to \mathbb{R}$ two simple integrable functions (resp. positive). Then functions $f + g$, αf ($\alpha \in \mathbb{R}$) are simple integrable (resp. positive) and we have relations:

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu; \int_X (\alpha f) d\mu = \alpha \int_X f d\mu.$$

Corollary 1. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \mathbb{R}$ two simple integrable functions (resp. positive) and $f \leq g$. Then $\int_X f d\mu \leq \int_X g d\mu$.

Corollary 2. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \mathbb{R}$ two simple functions with $|f| \leq g$ and g is integrable. Then f is integrable.

Proposition 4. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ a simple function with $f \geq 0$ μ-a.e. Then $\int_X f d\mu \geq 0$.

Corollary 3. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \mathbb{R}$ two simple integrable functions. Then the following assertions hold:

1. $f \leq g$ μ-a.e if and only if $\int_A f d\mu \leq \int_A g d\mu$, for any $A \in \mathcal{A}$
2. $f = g$ μ-a.e if and only if $\int_A f d\mu = \int_A g d\mu$, for any $A \in \mathcal{A}$.

Proposition 5. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ a simple, integrable function (resp. positive). Then for any $(A_n)_{n \geq 1} \subseteq \mathcal{A}$ mutually disjoint family and $A = \bigcup_{n \in \mathbb{N}} A_n$ we have $\int_A f d\mu = \sum_{n \geq 1} \int_{A_n} f d\mu$.

Corollary 4. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ a positive simple function. Then function $\nu_f(A) = \int_A f d\mu$, $A \in \mathcal{A}$ is a measure.

Definition 2. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}_+$ a positive measurable function. The number (finite or $+\infty$) $\sup\{\int_X h d\mu \mid 0 \leq h \leq f, h \text{ simple}\}$ is called integral of f on X and is denoted with $\int_X f d\mu$, $\int_X f(x) d\mu$, or $\int_X f$. We will say that f is integrable on X if $\int_X f d\mu < +\infty$. If $A \in \mathcal{A}$ we will put $\int_A f d\mu = \int_X f \cdot \varphi_A d\mu$ and we will say that f is integrable on A if $\int_A f d\mu < +\infty$.

Definition 3. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ an arbitrary measurable function. If $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < +\infty$ we will say that f has integral, and number (finite or $\pm\infty$) $\int_X f^+ d\mu - \int_X f^- d\mu$ is called integral of f on X and is denoted by $\int_X f d\mu$, $\int_X f(x) d\mu$, or $\int_X f$. If $\int_X f d\mu$ is finite we will say that f is integrable on X. If $A \in \mathcal{A}$ we will put $\int_A f d\mu = \int_X f \cdot \varphi_A d\mu$ if $f\varphi_A$ has integral, and we will say that f is integrable on A if $\int_A f d\mu$ is finite i.e. $\int_A f^+ d\mu < +\infty$ and $\int_A f^- d\mu < +\infty$.

Remark 1. 1. Let $f : X \to \overline{\mathbb{R}}$ a measurable function. If f is integrable on a set $A \in \mathcal{A}$ the integral $\int_A f d\mu$ is unique determinated by f and A.

2. A positive measurable function has integral always, but is not necessarily integrable (ex. the function $1_{\mathbb{R}}$).

3. An arbitrary measurable function can or can not to have integral (ex. function sign).

4. If $f : X \to \overline{\mathbb{R}}_+$ measurable. Then $\int_A f d\mu \geq 0$ for any $A \in \mathcal{A}$.

5. If $f : X \to \overline{\mathbb{R}}$ is a measurable function with $f = 0$ μ-a.e. then f is integrable and $\int_X f d\mu = 0$.

6. If $f : X \to \overline{\mathbb{R}}$ is measurable and $f \geq 0$ μ-a.e. then there exists $\int_X f d\mu$ and is positive.

7. Let $f : X \to \overline{\mathbb{R}}$ a measurable function. Then f is integrable on any set $A \subseteq X$ of null measure and $\int_A f d\mu = 0$.

8. If $f : X \to \overline{\mathbb{R}}$ is a positive, measurable function then

$$\int_A (-f) d\mu = -\int_A f d\mu \text{ for any } A \in \mathcal{A}.$$

9. Let $f, g : X \to \overline{\mathbb{R}}_+$ be two measurable functions. Then

$$\int_X (f + g) d\mu \leq \int_X f d\mu + \int_X g d\mu.$$

Proposition 6. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \overline{\mathbb{R}}$ two measurable functions. The following assertions hold:

1. $0 \leq f \leq g$ it follows that $\int_A f d\mu \leq \int_A g d\mu$, for any $A \in \mathcal{A}$.
2. $0 \leq f \leq g$ and g is integrable it follows that f is integrable.

Proposition 7. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ a measurable function. Then f is integrable if and only if $|f|$ is integrable.

Corollary 5. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ a measurable function. Then f is integrable if and only if there exists $g : X \to \overline{\mathbb{R}}$ positive, integrable such that $|f| \leq g$.

Proposition 8. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ an integrable function. Then f is integrable on any set $A \subset \mathcal{A}$.

Proposition 9. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ an integrable function. Then

$$\left|\int_A f d\mu\right| \leq \int_A |f| d\mu \leq \sup_{x \in A} |f(x)| \cdot \mu(A) \text{ for any } A \in \mathcal{A}.$$

Remark 2. 1. Let $(X, \mathcal{A}, \mu)$ be a space with finite measure and $f : X \to \mathbb{R}$ a measurable bounded function. Then f is integrable.

2. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ an integrable function. Then f finite μ-a.e.

(There exists measurable functions, finite μ-a.e but is not integrable, ex. the constant function 1 on $\mathbb{R}$).

Proposition 10. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \overline{\mathbb{R}}$ two positive measurable function. Then assertions hold:

1. $f \leq g$ μ-a.e it follows that $\int_X f d\mu \leq \int_X g d\mu$.
2. $f = g$ μ-a.e it follows that $\int_X f d\mu = \int_X g d\mu$.

Corollary 6. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \overline{\mathbb{R}}$ two functions such that f is measurable, g is integrable and $|f| \leq g$ μ-a.e. Then f is integrable.

Proposition 11. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ a measurable positive function. The following assertions hold:

1. $\int_X f d\mu = 0$ if and only if $f = 0$ μ-a.e.
2. $\int_A f d\mu = 0$ if and only if $f = 0$ μ-a.e on A, for any $A \in \mathcal{A}$.

Proposition 12. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ a measure function. Then the following assertions hold:

1. $(\forall)$ $\alpha \geq 0$ and $A : \{x \in X \mid |f(x)| \geq \alpha\} \Rightarrow \alpha\mu(A) \leq \int_A |f| d\mu \leq \int_X |f| d\mu$
2. $(\forall)$ $\beta \geq 0$ and $B := \{x \in X \mid |f(x)| \leq \beta\} \Rightarrow \int_B |f| d\mu \leq \beta\mu(B)$
3. $(\forall)$ $0 \leq \alpha \leq \beta$ and $C := \{x \in X \mid \alpha \leq |f(x)| \leq \beta\} \Rightarrow \alpha\mu(C) \leq \int_C |f| d\mu \leq \beta\mu(C)$.

Corollary 7. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ an integrable function. Then the following assertions hold:

1. $(\forall)$ $\varepsilon > 0$, $(\exists)$ $\alpha > 0$ such that $\mu(\{x \in X | f(x)| \geq \alpha\}) < \varepsilon$.
2. $(\forall)$ $\alpha > 0$, $\mu(\{x \in X \mid |f(x)| \geq \alpha\}) < \infty$.
3. $\mu(\{x \in X \mid |f(x)| = \infty\}) = 0$ i.e f finite μ-a.e.

3 Sequences of integrable functions

Proposition 13. Let $(X, \mathcal{A}, \mu)$ be a space with measure, $f : X \to \overline{\mathbb{R}}$ a simple integrable function. Then for any increasing sequence $(A_n)_{n \geq 1} \subseteq \mathcal{A}$ and $A = \bigcup_{n \geq 1} A_n$ we have $\int_A f d\mu = \lim_{n \to \infty} \int_{An} f d\mu$.

Theorem 1. (Beppo-Levi) (monotonous convergence theorem) Let $(X, \mathcal{A}, \mu)$ a space with measure, $f_n : X \to \overline{\mathbb{R}}_+$ $(n \geq 1)$ an increasing sequence of measurable functions and $f(x) = \lim_{n \to \infty} f_n(x)$, for any $x \in X$. Then we have

$$\lim_{n \to \infty} \int_A f_n d\mu = \int_A f d\mu \text{ for any } A \in \mathcal{A}.$$

Corollary 8. Let $(X, \mathcal{A}, \mu)$ be a space with measure $f_n : X \to \overline{\mathbb{R}}_+$ $(n \geq 1)$ a sequence of positive measurable functions. Then $f := \sum_{n \geq 1} f_n$ is a measurable function and $\int_A f d\mu = \sum_{n \geq 1} \int_A f_n d\mu$, for any $A \in \mathcal{A}$.

4 Algebraic operations with integrable functions

Let be $(X, \mathcal{A}, \mu)$ a space with measure, $f, g : X \to \overline{\mathbb{R}}$ two integrable functions and $\alpha, \beta \in \mathbb{R}$. Then function $\alpha f + \beta g$ is integrable and we have relation:

$$\int_A (\alpha f + \beta g) d\mu = \alpha \int_A f d\mu + \beta \int g d\mu \text{ for any } A \in \mathcal{A}.$$

Proposition 14. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, g : X \to \overline{\mathbb{R}}$ two integrable functions. Then the following assertions hold:

1. $f \leq g$ μ-a.e if and only if $\int_A f d\mu \leq \int_A g d\mu$ for any $A \in \mathcal{A}$.
2. $f = g$ μ-a.e if and only if $\int_A f d\mu = \int_A g d\mu$ for any $A \in \mathcal{A}$.

Corollary 9. Let $(X, \mathcal{A}, \mu)$ be a space with complete measure and $f, g : X \to \overline{\mathbb{R}}$ two functions such that $f = g$ μ-a.e. If f is integrable then g is integrable and $\int_A f d\mu = \int_A g d\mu$, for any $A \in \mathcal{A}$.

Theorem 2. (Fatou) Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f_n : X \to \overline{\mathbb{R}}$, $(n \geq 1)$ a sequence of positive measurable functions. Then we have that

$$\int_A \underline{\lim}_{n \to \infty} f_n d\mu \leq \underline{\lim}_{n \to \infty} \int_A f_n d\mu$$

for any $A \in \mathcal{A}$.

Corollary 10. Let $(X, \mathcal{A}, \mu)$ be a space with complete measure and $f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$ a sequence of positive measurable functions, such that $f_n \overset{\mu-a.e}{\longrightarrow} f$, $f : X \to \overline{\mathbb{R}}$, with $f_n \leq f$ μ-a.e for any $n \geq 0$. Then $\lim_{n \to \infty} \int_A f_n d\mu = \int_A f d\mu$ for any $A \in \mathcal{A}$.

Remark 3. In the above theorem we have not necessarily equality.

Example. We consider space with measure $((0,1), \mathcal{L}\,|_{(0,1)}, \mu)$ and $f_n : (0,1) \to \mathbb{R}$, where $f_n = n\varphi_{(0,1/n)}$ $(n \geq 1)$. Then $f_n \overset{s}{\to} 0$, hence $\int_{(0,1)} \underline{\lim}_{n\to 0} f_n d\mu = 0$. On the other hand we have $\int_{(0,1)} f_n d\mu = 1$ for any $n \geq 1$, hence $\lim_{n\to\infty} \int_{(0,1)} f_n d\mu = 1$. Hence inequality from above theorem is strictly.

Remark 4. The condition as functions $(f_n)_{n\geq 1}$ to be positive from above Theorem is essentially. If we consider space with measure $(\mathcal{R}, \mathcal{L}, \mu)$ and $f_n : \mathbb{R} \to \mathbb{R}$ where $f_n = -\frac{1}{n} \cdot \varphi_{[0,n]}$, $(n \geq 1)$. Then $f_n \overset{s}{\to} 0$ and we have $\int_{\mathbb{R}} f_n d\mu = -\frac{1}{n} \int_{[0,n]} d\mu = -1$, hence

$$\int_{\mathbb{R}} \underline{\lim}_{n\to\infty} f_n d\mu > \underline{\lim}_{n\to\infty} \int_{\mathbb{R}} f_n d\mu.$$

Theorem 3. (Lebesgue) (dominated convergence theorem)
Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f, f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$ measurable, such that $f_n \overset{\mu-a.e}{\longrightarrow} f$ and there exists, $g : X \to \overline{\mathbb{R}}$ integrable with $|f_n| \leq g$ μ-a.e for any $n \geq 1$. Then f is integrable and $\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu$ for any $A \in \mathcal{A}$.

Theorem 4. (Integral absolute continuity theorem)
Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \overline{\mathbb{R}}$ an integrable function. Then $\lim_{\mu(A)\to 0} \int_A f d\mu = 0$ i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\int_A f d\mu| \leq \varepsilon$ for any $A \in \mathcal{A}, \mu(A) < \delta$.

Corollary 11. Let be $(X, \mathcal{A}, \mu)$ a space with measure and $f : X \to \overline{\mathbb{R}}$ an integrable function. Then for any family $(A_n)_{n\geq 1} \subseteq \mathcal{A}$, which for $\mu(A_n) \to 0$ it follows that $\int_{A_n} |f| d\mu \to 0$ $(n \to \infty)$.

Corollary 12. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f_n : X \to \overline{\mathbb{R}}_+$ $(n \geq 1)$ a monotonous increasing of positive measurable functions such that $\sup_{n\geq 1} \int_X f_n < \infty$. Then there exists $f : X \to \overline{\mathbb{R}}$ integrable such that $f_n \overset{\mu-a.e}{\longrightarrow} f$ and $\int_X f_n d\mu \to \int_X f d\mu$.

Corollary 13. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f_n : X \to \overline{\mathbb{R}}_+$ $(n \geq 1)$ a decreasing sequence of positive, measurable functions such that $\int_X f_1 d\mu < \infty$. Then function $f(x) = \lim_{n\to\infty} f_n(x)$ for any $x \in X$ is integrable and $\int_X f_n d\mu \overset{n\to\infty}{\longrightarrow} \int_X f$.

5 Particular classes of integrable functions

Proposition 15. Let $f : [a,b] \to \mathbb{R}$ a function. Then f is Riemann integrable if and only if f is bounded and the set $D_f = \{x \in [a,b] \mid f \text{ is not continuous in } x\}$ is Lebesgue negligible.

Theorem 5. Let $f : [a,b] \to \mathbb{R}$ a Riemann integrable function. Then f is Lebesgue integrable and two integrals coincide.

Therefore Lebesgue integral is strict extension of Riemann integral (Dirichlet function example).

1. A set $f : [-1,1] \to \mathbb{R}$ Riemann function defined as:

$$f(x) = \begin{cases} 0, & \text{if } x \notin Q \text{ or } x = 0 \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ (irreducible fraction) } (q \neq 0) \end{cases}$$

f is discontinuous at any rational point $x = \frac{p}{q}$ $(q \neq 0)$ and continuous at any irrational point and f is bounded.

$A = Q \cap [-1,1] \setminus \{0\}$ the set of discontinuity points of f. Hence $\mu(A) = 0$.

Therefore f is Riemann integrable on $[-1,1]$, hence Lebesgue integrable $\int_{-1}^{1} f(x)dx = \int_{[-1,1]} f d\mu = 0$, because $f = 0$ μ-a.e.

2. Let $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \in [0,1] \\ 0, & x = 0. \end{cases}$ To show that f is Lebesgue integrable and integral to calculate.

f is not Riemann integrable because is not bounded. f is Lebesgue measurable because is λ-a.e. continuous. Let $f_n(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \in \left(\frac{1}{n}, 1\right] \\ 0, & x \in \left[0, \frac{1}{n}\right] \end{cases}$, $n \geq 2$. f_n is continuous, for any $n \geq 2 \Rightarrow f_n$ Riemann integrable (resp. Lebesgue integrable).

$$\begin{aligned} \int_{[0,1]} f_n d\mu &= \int f_n dx = \int_0^{1/n} 0dx + \int_{1/n}^{1} \frac{1}{\sqrt{x}} dx = \\ &= 2x^{1/2}\Big|_{1/n}^{1} = 2\left(1 - \frac{1}{\sqrt{n}}\right). \end{aligned}$$

We have that $\lim_{n\to\infty} f_n = f$.
Then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu = 2$.

Another remarkable class of Lebesgue integrable functions is given by uncompact absolutely integrable functions on Riemann improper sense on intervals of $\mathbb{R}$ (resp. $\mathbb{R}^n$)

Let $f : \mathbb{R} \to \mathbb{R}$ such that f Riemann integrable on $[a,b]$ for any $a, b \in \mathbb{R}$ and there exists next limit

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b |f(t)| dt \in \mathbb{R}.$$

Under these conditions is said that integral $\int_{-\infty}^{+\infty} f(t)dt$ is absolutely convergent or that $|f|$ is Riemann integrable in the improper sense on $\mathbb{R}$.

Putting $f_n = |f| \cdot \lambda_{[-n,n]}$, $\forall\, n \in \mathbb{N}$, then we get $f_n \uparrow |f|$. Hence f_n is Riemann integrable (resp. Lebesgue) and $\int_{\mathbb{R}} f_n d\mu = \int_{-n}^{n} |f(t)| dt$. Since there exists

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b |f(t)| dt \in \mathbb{R}.$$

it follows that from Beppo-Levi theorem the function $|f|$ is Lebesgue integrable on $\mathbb{R}$.

Since $f \cdot \lambda_{[-n,n]}$, $n \geq 1$ are Riemann integrable, hence Lebesgue measurable then we get f Lebesgue measurable. Hence

$$\int_{\mathbb{R}} f d\mu = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(t)dt = \int_{-\infty}^{+\infty} f(t)dt.$$

Example. Let $\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}_+ \ \mu(A) = \begin{cases} \text{card } A, & A \text{ finite} \\ +\infty & A \text{ infinite.} \end{cases}$ Then we obtain the space with measure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ and $f : \mathbb{N} \to \mathbb{R}$. Obviously f is measurable and $f = \sum_{n \in \mathbb{N}} f(n)\lambda_{\{n\}}$. f will be integrable if and only if $\sum_{n \in \mathbb{N}} |f(n)| < +\infty$, i.e. series $\sum_{m \in \mathbb{N}} f(n)$ is absolutely convergent.

Remark 5. Theory of absolutely convergent series is in fact a theory of integrability.

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