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ON PARAMETRIC QUASIVARIATIONAL INEQUALITIES

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ABSTRACT

A sequence of parametric quasivariational inequalities related to a "limit" quasivariational inequality is considered. We use our technique from [2], *to provide a different proof that one given in* [1] *in order to obtain the stability of solutions, via topological pseudomonotonicity* (*in the sense of Brézis*) *of the "limit" operator.*

Keywords: quasivariational inequality, topological pseudomonotonicity, hemicontinuity, Mosco convergence, solutions set

1 Introduction

For the motivation and significance of a variational inequality one can consult for example [4, 6, 7].

Let X be a Hilbert space and let us consider, for any $n \in \mathbb{N}$, the following parametric quasivariational inequality:

$(QVI)_n$ Find an element $a_n \in D_n(a_n)$ such that

$$\langle T_n a_n, b - a_n \rangle \geq 0, \ \forall b \in D_n(a_n),$$

where $D_n(a_n)$ is a nonempty, closed, convex subset of X and $T_n : X \to X$ is a given operator.

Along with $(QVI)_n$ we consider the so-called "limit" quasivariational inequality:

(QVI) Find an element $a \in D(a)$ such that

$$\langle Ta, b - a_n \rangle \geq 0, \ \forall b \in D(a),$$

where $D(a) \subseteq X$ is nonempty closed convex and and $T : X \to X$ is a given operator.

Denote by $S(n)$ the set of the solutions of $(QVI)_n$ for a fixed n and by $S(\infty)$ the set of the solutions for problem (QVI).

Our purpose is to get closer [2] and [1], motivated and inspired by the hypotheses involved in the main results of the two mentioned articles.

2 Main part

In [1] a control space U as a Hilbert space is considered. For any control parameter $u \in U$, a bit more general problem then $(QVI)_u$ was studied. Precisely, for $\varphi \equiv 0$ in [1], the quasivariational inequality is the following:

$(QVI)_u$ Find an element $y_u \in D(y_u)$ such that

$$\langle Ay_u - Bu, y - y_u \rangle \geq 0, \forall y \in D(y_u).$$

Let us shortly recall their setting. The Hilbert spaces V is such that the embedding $V \subset X$ is continuous, compact and dense. By V^* is denoted the topological dual of V. The operators $A : V \to V^*$ and $B : U \to V^*$ are given. The set-valued map $D : V \to 2^V$ has nonempty, closed, convex values.

Let $u \in U$ and $(u_n)_{n \in \mathbb{N}}$ be weakly convergent to u in U. Along with $(QVI)_u$ consider the following parametric problems:

$(QVI)_{u_n}$ Find an element $y_{u_n} \in D(y_{u_n})$ such that

$$\langle Ay_{u_n} - Bu_n, y - y_{u_n} \rangle \geq 0, \forall y \in D(y_{u_n}).$$

Define the set-valued map $S : u \longmapsto S(u)$ the solution map of problems $(QVI)_{u_n}$ and $(QVI)_u$. To overlap in our framework one should regard $S(n) = S(u_n)$ and $S(\infty) = S(u)$.

In [1] it was established the weak sequential closedness of S, i.e. if $(y_n)_{n \in \mathbb{N}}$ is a sequence with $y_n \in S(n)$ such that $y_n \rightharpoonup y$ (weakly) in V then $y \in S(\infty)$.

Let us remind some usual definitions (see [6, 7]). A single operator $A : V \to V^*$ is said to be hemicontinuous if it is weakly continuous on all segments, i.e. $\lim_{\lambda \to 0} \langle A(y + \lambda z), w \rangle = \langle A(y), w \rangle$, for all $y, z, w \in V$. A is said to be strongly monotone if there exists a constant $L > 0$ such that $\langle A(y) - A(z), y - z\rangle \geq L\|y - z\|^2$, for all $y, z \in V$. A is topologically pseudomonotone if for each sequence $(y_n)_{n\in\mathbb{N}} \subset V$ with $y_n \rightharpoonup y$ in V, $\liminf_n \langle A(y_n), y - y_n\rangle \geq 0$ imply $\limsup_n \langle A(y_n), z - y_n\rangle \leq \langle A(y), z - y\rangle$, for all $z \in V$.

In our previous works [2, 3] we use the following condition:

($\mathbf{C}$) For each sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \in S(n)$ if $a_n \rightharpoonup a$, $b_n \in X$, $b \in X$, and $b_n \to b$, then

$$\liminf_n \left(\langle Ta_n, b - a_n\rangle - \langle T_n a_n, b_n - a_n\rangle\right) \geq 0.$$

Following the classical article [5], the sequence of sets $(D_n)_{n\in\mathbf{N}}$ Mosco converges to D if:

1. for every subsequence $(n_k)_{k\in\mathbf{N}}$, $a_{n_k} \in D_{n_k}$ and $a_{n_k} \rightharpoonup a$ imply $a \in D$;
2. for every $b \in D$, there exists $(b_n)_{n\in\mathbf{N}}$, such that $b_n \in D_n$ and $b_n \to b$.

To state our result we remind the following (see [2], Theorem 1)

Corollary 1. *Let X be a Hilbert space. Let D_n be nonempty sets of X, $n \in \mathbb{N}$. Suppose that $S(n) \neq \emptyset$, for each $n \in \mathbb{N}$, and the following conditions hold:*

i) D_n Mosco converges to D;

ii) T_n and T verify ($\mathbf{C}$);

iii) T is topologically pseudomonotone.

Then, for each sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in S(n)$, $a_n \rightharpoonup a$ implies $a \in S(\infty)$.

Related to Theorem 2 from [1] we are able to formulate the following result:

Theorem 1. *Let $u \in U$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence weakly convergent to u in U. Assume the following:*

(i) *$D(y)$ is closed and convex for all $y \in V$;*

(ii) *For every $(y_n)_{n\in\mathbb{N}}$ such that y_n weakly converges to y in V, then $D(y_n)$ Mosco converges to $D(y)$;*

(iii) *$A : V \to V^*$ is hemicontinuous, strongly monotone and bounded;*

(iv) *$B : U \to V^*$ is a compact operator.*

Then, $u \longmapsto S(u)$ is weakly closed, i.e. for each sequence $(y_n)_{n\in\mathbb{N}}$ of solutions to $(QVI)_{u_n}$, $y_n \rightharpoonup y$ imply $y \in S(u)$.

Proof. Let us check the hypotheses from Corollary 1.

Let us define the operators T and T_n, $n \in \mathbb{N}$ by $Ty = Ay - Bu$ and $T_n y = Ay - Bu_n$. In order to check our condition ($\mathbf{C}$) we evaluate the difference

$$\begin{aligned}&\langle Ty_n, z - y_n\rangle - \langle T_n y_n, z - y_n\rangle \\ &\quad = \langle Ay_n, z - y_n\rangle - \langle Ay_n, z_n - y_n\rangle \\ &\quad -[\langle Bu, z - y_n\rangle - \langle Bu_n, z_n - y_n\rangle].\end{aligned}$$

Let $z_n \to z$ and $y_n \rightharpoonup y$, where $\to$ and $\rightharpoonup$ denote the usual strong and weak convergence, respectively. Since B is compact, $(Bu_n)_{n\in\mathbb{N}}$ is strongly convergent to Bu in V^*, therefore the last bracket tends to 0. Since A is supposed to be bounded the same happens to the remaining terms.

The operator A is supposed to be hemicontinuous and strongly monotone, therefore is topologically pseudomonotone (see [6, 7]). By the same hypothesis (iii), A is also coercive so $S(n) \neq \emptyset$, for all $n \in \mathbb{N}$. Using the definition, $T = A - Bu$ is also topologically pseudomonotone.

In conclusion, by our Corollary 1, if $(y_n)_{n\in\mathbb{N}}$ is a sequence with $y_n \in S(n)$ such that $y_n \rightharpoonup y$ in V then $y \in S(\infty)$. □

A direct comparison between these two results, Assertion 2. in Theorem 2 in [1] and our Corollary 1, make us wonder whether or not the hypothesis for the operator A is redundant. Precisely, (iii) from Theorem 2 in [1] requires that for every $(z_n)_{n\in\mathbb{N}}$ strongly convergent to z in V and $(y_n)_{n\in\mathbb{N}}$ weakly convergent to y in V, $\langle Az, y - z\rangle \leq \liminf_n \langle Az_n, y_n - z_n\rangle$. But this property is not essential in none of the two assertions.

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