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LEBESGUE SETS IMMEASURABLE EXISTENCE

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ABSTRACT

It is well known that the notion of measure and integral were released early enough in close connection with practical problems of measuring of geometric figures. Notion of measure was outlined in the early 20th century through H. Lebesgue's research, founder of the modern theory of measure and integral. It was developed concurrently a technique of integration of functions. Gradually it was formed a specific area today called the measure and integral theory. Essential contributions to building this theory was made by a large number of mathematicians: C. Carathodory, J. Radon, O. Nikodym, S. Bochner, J. Pettis, P. Halmos and many others. In the following we present several abstract sets, classes of sets. There exists the sets which are not Lebesgue measurable and the sets which are Lebesgue measurable but are not Borel measurable. Hence $\mathcal{B} \subset \mathcal{L} \subset \mathcal{P}(X)$.

Keywords: *σ-algebra, class of equivalence, Borel measurable sets, Lebesgue measurable sets, immeasurable Lebesgue sets.*

1 Lebesgue measure on $\mathbb{R}$

Let $\mathbb{R}$ be the set of real numbers. We denote by

$$\mathcal{S} = \{[a, b) \mid a \in \mathbb{R},\ b \in \mathbb{R}\}.$$

We denote by μ the function $\mu : \mathcal{S} \to \mathbb{R}_+$, defined by $\mu([a, b)) = b - a$.

Definition 1. It is called Lebesgue outer measure the function

$$\mu^* : \left\{A \subset \mathbb{R} \mid (\exists)\ (E_n)_{n\in\mathbb{N}} \subset \mathcal{S},\ \bigcup_{n\in\mathbb{N}} E_n \supset A\right\} \to \mathbb{R}_+$$

defined by

$$\mu^*(A) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(E_n) \mid \bigcup_{n\in\mathbb{N}} E_n \supset A,\ (E_n)_{n\in\mathbb{N}} \in \mathcal{S}\right\}$$

Remark 1. Since $\mathbb{R} = \bigcup_n [-n, n) = \bigcup_{n\in\mathbb{Z}} [n, n+1)$ we have $\{A \subset \mathbb{R} \mid (\exists)\,(E_n)_{n\in\mathbb{N}} \subset \mathcal{S},\ \bigcup_n E_n \supset A\} = \mathcal{P}(\mathbb{R})$. Hence $\mu^* : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+$.

Theorem 1. The Lebesgue outer measure holds the following properties:

1. $\mu^*(\phi) = 0$
2. $A \subset B \subset \mathbb{R} \Rightarrow \mu^*(A) \leq \mu^*(B)$
3. $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}) \Rightarrow \mu^*(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$
4. $A \subset \mathbb{R}, t \in \mathbb{R} \Rightarrow \mu^*(A + t) = \mu^*(A)$
5. $\mu^* \mid \mathcal{S} = \mu$.

Definition 2. A subset $E \subset \mathbb{R}$ is called Lebesgue measurable if the following equality holds:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \mathcal{C}_E),\ \text{for any}\ A \in \mathcal{P}(\mathbb{R})$$

Remark 2. Since μ^* is increasing the equality is equivalent by inequality $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap CE)$. Hence equality is not trivial only for $A \in \mathcal{P}(\mathbb{R})$ with $\mu^*(A) < +\infty$.

We denote by $\mathcal{L} = \{E \in \mathcal{P}(\mathbb{R}) \mid E$ is Lebesgue measurable$\}$.

Theorem 2. For the couple $(\mathcal{L}, \mu^*)$ the following assertions hold:

1. $E, F \in \mathcal{L}$ we get $E \cup F \in \mathcal{L}$
2. $E, F \in \mathcal{L}$ we get $F \setminus E \in \mathcal{L}$
3. $(E_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ we get $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{L}$
4. $\mu^* \mid \mathcal{L}$ is a positive measure
5. $E \in \mathcal{P}(\mathbb{R})$, $\mu^*(E) = 0$, $F \subset E$ we get $F \in \mathcal{L}$ and $\mu^*(F) = 0$
6. $\mathcal{S} \subset \mathcal{L}$.
7. for any $E \in \mathcal{L}$ we get $\{x+t \mid x \in E\} = E+t \in \mathcal{L}$, for any $t \in \mathbb{R}$.

Conclusion 1. From above Theorem it follows that the Lebesgue measurable sets $\mathcal{L}$ form a σ-algebra which includes $\mathcal{S}$ and $\mu^* \mid_{\mathcal{L}}$ is a positive measure.

Remark 3. Restriction of μ^* to $\mathcal{L}$ is called induced measure by μ^* and was noted by $\overline{\mu}$.

Remark 4. The set $\mathcal{S}$ and the function μ have the following properties:

1. $E, F \in \mathcal{S} \Rightarrow E \cap F \in \mathcal{S}$
2. $E, F \in \mathcal{S} \Rightarrow E - F = \bigcup_{k=1}^{p} E_k$, $(E_k)_{1\leq k\leq p} \subset \mathcal{S}$, $E_i \cap E_j = \phi$.
3. $F \in \mathcal{S}$, $F = F_1 \cup F_2$, $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{S}$, $F_1 \cap F_2 = \phi$ it follows that $\mu(F) = \mu(F_1) + \mu(F_2)$.

2 Special properties of Lebesgue measure and measurability

We denote by μ restriction of outer Lebesgue measure to class of Lebesgue measurable sets and will call μ Lebesgue measure. We denote by $\mathcal{L}$ the sets Lebesgue measurable.

We denote by $\mathcal{B}$ the Borel sets on $\mathbb{R}$.

Any Borel set is Lebesgue measurable.

Lebesgue measure is only measure σ-finite on $\mathcal{B}$ whose restriction to $\mathcal{S}$ is the length intervals.

For any subset of $\mathbb{R}$ with outer Lebesgue measure finite there exists a Borel subset which contains and which has the same outer measure.

Any Lebesgue measurable set is reunion of a Borel set and a subset a Borel set of null Lebesgue measure.

For any Lebesgue measurable subset A of $\mathbb{R}$ with outer Lebesgue measure finite and for any $\varepsilon > 0$ there exists a finite reunion of intervals from $\mathcal{S}$ which differ to A whose outer Lebesgue measure is smaller than ε.

Lebesgue measure coincide with outer measures induced by restrictions of Lebesgue measure to $\mathcal{I}(\mathcal{S})$, $\mathcal{B}, \mathcal{L}$ we have

$$\mu^*(A) = \inf\{\mu(E) \mid A \subset E, E \in \mathcal{B}\} = \inf\{\mu(M) \mid A \subset M, M \in \mathcal{L}\}.$$

Theorem 3. Let $\mathcal{D}_{\mathbb{R}}$ be topology of $\mathbb{R}$, $\mathcal{F}_{\mathbb{R}}$ be the closed sets of $\mathbb{R}$,

$$\mathcal{I}_1 = \{(a,b) \mid a,b \in \mathbb{R}\}; \quad \mathcal{I}_2 = \{(a,b] \mid a,b \in \mathbb{R}\},$$
$$\mathcal{I}_3 = \{[a,b] \mid a,b \in \mathbb{R}\}, \quad \mathcal{I}_4 = \{[a,+\infty) \mid a \in \mathbb{R}\},$$
$$\mathcal{I}_5 = \{(a,+\infty) \mid a \in \mathbb{R}\}, \quad \mathcal{I}_6 = \{(-\infty,a] \mid a \in \mathbb{R}\},$$
$$\mathcal{I}_7 = \{(-\infty,a) \mid a \in \mathbb{R}\}.$$

Then $\mathcal{B} = \sigma(\mathcal{D}_{\mathbb{R}}) = \sigma(\mathcal{F}_{\mathbb{R}}) = \sigma(\mathcal{I}_k)$ for any k, $1 \leq k \leq 7$.

Corollary 1. $\mathbb{R}$ is a Borel set, $\{x\}$ is a Borel set for any $x \in \mathbb{R}$ and any subset at most countable of $\mathbb{R}$ is Borel set.

The outer Lebesgue measure is $\mu^*(A) = \inf\{\mu(D) \mid A \subset D, D \in \mathcal{D}_{\mathbb{R}}\}$, for any $A \subset \mathbb{R}$.

Any at most countable subset of $\mathbb{R}$ has Lebesgue measure null.

There exists subset or $\mathbb{R}$ which is not Lebesgue measurable.

Example. For any $M \subset [0,1]$, $M \in \mathcal{L}$, $\mu(M) > 0$ there exists $A \subset M$ such that $A \notin \mathcal{L}$.

Proof. We will define an equivalence relation:

$$x \sim y \text{ if and only if } x - y \in Q.$$

We consider ξ the set of distinct equivalence class determined by the elements of M. We choice in each equivalence class from ξ an element from M and we denote by A the set all these elements.

Obviously $M \subset \bigcup_{t\in Q} A + t$, since each element of M belongs to equivalence class determined by himself. On the other hand $(A + t_1) \cap (A + t_2) = 0$ for $t_1 \neq t_2, t_1, t_2 \in Q$ since otherwise $a_1 + t_1 = a_2 + t_2$, $a_i \in A$, $i = 1, 2$ hence $a_1 - a_2 = t_1 - t_2 \in Q$ i.e. $a_1 \sim a_2$ contradiction with construction of A (the elements of A are in distinct equivalence class hence is not equivalent).

We suppose that A is Lebesgue measurable, hence $A \in \mathcal{L}$. From invariance to translations of Lebesgue measure we deduce $A + t \in \mathcal{L}$ and $\mu(A) = \mu(A + t)$, for any $t \in Q$. Since μ is σ-additive and monoton it follows that

$$\mu(M) \leq \sum_{t\in Q} \mu(A+t) = \sum_{t\in Q} \mu(A)$$

Since $\mu(M) > 0$ it follows that $\mu(A) > 0$, and we have

$$\bigcup_{t\in[0,1]\cap Q} A+t \subset \bigcup_{t\in[0,1]\cap Q} M$$
$$+t \subset \bigcup_{t\in[0,1]\cap Q} [0,1]+t \subseteq [0,2].$$

We deduce then

$$\sum_{t\in[0,1]\cap Q} \mu(A+t) = \sum_{t\in[0,1]\cap Q} \mu(A)$$
$$= \mu\left(\bigcup_{t\in[0,1]\cap Q} A+t\right) \leq 2.$$

Above inequality occurs if $\mu(A) = 0$ contradiction with $\mu(A) > 0$. Hence $A \notin \mathcal{L}$. □

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