Homomorphic embeddings in \(n\)-groups

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ABSTRACT

We prove that an cancellative \(n\)-groupoid \(\mathcal{A}\) can be homotopic embedded in an \(n\)-group if and only if in \(\mathcal{A}\) are satisfied all \(n\)-ary Malcev conditions. Now we shall prove that in the presence of associative law we obtain homomorphic embeddings. Furthermore, if \(\mathcal{A}\) has a lateral identity a such embeddings is assured by a subset of \(n\)-ary Malcev conditions - unary Malcev conditions.

**Keywords:** cancellation law, covering semigroup, homotopic embedding, \(n\)-ary Malcev conditions, \(n\)-groupoid, unary Malcev conditions.

We prove that an cancellative \(n\)-groupoid \(\mathcal{A}\) can be homotopic embedded in an \(n\)-group if and only if in \(\mathcal{A}\) are satisfied all \(n\)-ary Malcev conditions.

Now we shall prove that in the presence of associative law we obtain homomorphic embeddings. Furthermore, if \(\mathcal{A}\) has a lateral identity a such embeddings is assured by a subset of \(n\)-ary Malcev conditions - unary Malcev conditions.

For an abbreviation we shall use the following notations (see [1]):

\[
(x_1, x_2, \ldots, x_n) = x_1^n,
\]

respectively \(x^n\) if

\[
x_1 = x_2 = \cdots = x_n = x.
\]

Let \(\mathcal{A} = (A, \alpha)\) be an \(n\)-groupoid (i.e \(\alpha : A^n \to A\)). If \(\alpha\) satisfies the associative law

\[
\alpha(\alpha(x_1^n), x_2^{2n-1}) = \alpha(x_1, \alpha(x_{i+1}^{n+i}, x_2^{2n-1}))
\]

for \(i = 1, 2, \ldots, n - 1\) and for all \(x_1, \ldots, x_{2n-1}\) in \(A\) then \(\mathcal{A}\) is an \(n\)-semigroup.

The sequence \(a_1^{n-1}\) is a lateral identity in the \(n\)-groupoid \(\mathcal{A}\) if

\[
\alpha(a_1^{n-1}, x) = \alpha(x, a_1^{n-1}) = x
\]

for all \(x\) in \(A\).

The following laws, wich may of may not hold in a given \(n\)-groupoid \(\mathcal{A}\), are known as left and right cancellation laws, respectively,

\[
\alpha(u_1^{-1}, x) = \alpha(u_1^{-1}, y) \Rightarrow x = y
\]

\[
\alpha(x, u_1^{n-1}) = \alpha(y, u_1^{n-1}) \Rightarrow x = y
\]

An \(n\)-groupoid \(\mathcal{A}\) is a cancellation \(n\)-groupoid if

\[
\alpha(u_1^{i-1}, x, u_2^{i+1}) = \alpha(u_1^{i-1}, y, u_2^{i+1}) \Rightarrow x = y
\]

for \(i = 1, 2, \ldots, n - 1\).

In [5] was proved that an \(n\)-semigroup wich is left and right cancellative is a cancellation \(n\)-semigroup.

An important concept in the theory of \(n\)-semigroups is that of a covering semigroup.

**Definition 1.** (see [5]) A binary \(\overline{\mathcal{A}} = (\overline{A}, \cdot)\) semigroup is said to be a covering semigroup of an \(n\)-semigroup \(\mathcal{A} = (A, \alpha)\) provided \(\overline{\mathcal{A}}\) has the following properties:

- the set \(A\) is a generating subset of \(\overline{A}\);
- \(\alpha(a_1^n) = a_1 \cdot a_2 \ldots a_n\) for all \(a_1, \ldots, a_n \in A\).

Generalizing an result from [5] we have

**Theorem 1.** Every cancellation \(n\)-semigroup has a cancellation covering semigroup.

**Outline of proof.** Let \(\mathcal{A} = (A, \alpha)\) be an cancellation \(n\)-semigroup. Denote by \(S' = (S', \cdot)\) the free semigroup with identity generated by the set \(A\). Let us consider the binary relation \(\pi \subseteq S'^2\) defined by: \(ss'\) iff

1. there exist \(s_1, s_2, s_3 \in S'\) such that \(\lambda(s_2) = n\) (where \(\lambda(s_2)\) is the length of \(s_2\), \(s = s_1s_2s_3\) and \(s' = s_1\alpha(s_2)\) or

2. \(\lambda(s) = \lambda(s') < n\) and there is a \(s'' \in S'\) with \(\lambda(s'') = n - \lambda(s)\) such that \(\alpha(ss') = \alpha(ss'')\), or
3. \( s = 1 \) (the identity of \( S' \)), \( \lambda(s') = n - 1 \) and 
\( \alpha(s', a) = a \) for some \( a \in A \).

Denote by \( \rho \) the equivalence on \( S' \) generated by \( \pi \). 
Then \( \rho \) is a congruence on \( S' \) and \( S'/\rho \) is a cancellation 
covering semigroup of \( A \).

It is easy to prove the following

**Lemma 1.** Let \( \bar{A} \) a covering semigroup of the \( n \)-semigroup \( A \). If \( \bar{A} \) can be homomorphic embedded in 
a group then \( A \) can be homomorphic embedded in a \( n \)-group.

**Theorem 2.** A cancellation \( n \)-semigroup \( A = (A, \alpha) \) without lateral identities can be homomorphic embedded 
in a \( n \)-group iff in \( A \) are satisfied all \( n \)-ary Malcev conditions.

Proof. Suppose that \( A \) can be homomorphical embedded in an \( n \)-group \( G \). All \( n \)-ary Malcev conditions 
satisfied in \( G \). Consequently, these conditions are satisfied in \( A \).

Conversely, assume that all \( n \)-ary Malcev conditions 
satisfied in \( A \). By Lemma 1 it is sufficient to prove that the covering semigroup \( S'(A)/\rho \) is 
malcev embeddable in a binary group. \( A \) being without lateral identities, \( [1] \) is a prime unit in 
\( S'(A)/\rho \). Therefore it is sufficient to prove that the 
semigroup \( S(A)/\rho = (S'(A)/\rho \setminus \{[1]\}, \cdot) \) is embed- 
dable in a group. There exists such an embedding if in 
\( S(A)/\rho \) are satisfied all binary Malcev conditions (see [3]). Since \( \{(a) \mid a \in A \} \) is a generating set of \( S(A)/\rho \) it is 
sufficient (see [3]) to consider only Malcev conditions 
according the table

<table>
<thead>
<tr>
<th>( L_i )</th>
<th>( \bar{L}_i )</th>
<th>( R_i )</th>
<th>( \bar{R}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a_1][s_1] )</td>
<td>([a_1][s_1] )</td>
<td>([w_1][t_1] )</td>
<td>([\bar{w}_1][\bar{t}_1] )</td>
</tr>
<tr>
<td>([u_1][s_1] )</td>
<td>([u_1][s_1] )</td>
<td>([\bar{w}_1][t_1] )</td>
<td>([\bar{w}_1][\bar{t}_1] )</td>
</tr>
</tbody>
</table>

Let \( I \) be a Malcev sequence and \( \sigma(I) \) the corresponding 
system of equalities. Adjoinining the closing 
equality to \( \sigma(I) \) we obtain the system \( \tilde{\sigma}(I) \). To each 
equality of \( \tilde{\sigma}(I) \) we assign a tag - the corresponding 
pair of symbols of \( I \).

**Example.** Let \( I = R_1L_1R_2L_2R_3[\bar{T}_2][\bar{T}_3][\bar{T}_2][\bar{T}_1][\bar{T}_1] \).

The tagged system \( \tilde{\sigma}(I) \) is

\[
\begin{align*}
(R_1L_1) & \ [w_1][\bar{a}_1] = [u_1][s_1] \\
(L_1R_2) & \ [a_1][s_1] = [w_2][t_2] \\
(R_2L_2) & \ [w_2][\bar{a}_2] = [u_2][s_2] \\
(L_2R_3) & \ [a_2][s_2] = [w_3][t_3] \\
(R_3L_2) & \ [w_3][\bar{a}_3] = [a_2][s_2] \\
([T_2][\bar{T}_3]) & \ [u_2][\bar{s}_2] = [\bar{w}_3][\bar{a}_3] \\
([\bar{T}_3][\bar{T}_2]) & \ [\bar{w}_3][\bar{a}_3] = [\bar{w}_2][\bar{a}_2] \\
([\bar{T}_2][\bar{T}_1]) & \ [\bar{w}_2][t_2] = [a_1][s_1] \\
([T_1][\bar{T}_1]) & \ [u_1][s_1] = [\bar{w}_1][\bar{a}_1] \\
([\bar{T}_1][R_1]) & \ [w_1][t_1] = [u_1][t_1] \text{ (the closing equality)}
\end{align*}
\]

From the definition of the congruence relation \( \rho \) it follows:

- if \( [x] = [y] \) then \( \lambda(x) \equiv \lambda(y) (\text{mod } n - 1) \), where 
  \( \lambda(x) \) is the length of \( x \);
- in each class \( [x] \) there is an element \( x' \) with 
  \( \lambda(x') \leq n - 1 \).

Consequently, we can suppose that in the table 1 
each representative has the length \( \leq n - 1 \).

Now we construct a new system of equalities \( \tilde{\sigma} \) in 
which member has the length \( \equiv 1 (\text{mod } n - 1) \). Let \( a \) be 
an element of \( A \).

1. If \( L_1 \) is the first symbol of \( I \),
   \( (L_1\cdot) \ [a_1][s_1] = [x][y] \)
   we choose \( 0 \leq j_1 \leq n - 1 \) such that \( \lambda(s_1) + j_1 \equiv 0 \).

2. If \( R_1 \) is the first symbol of \( I \),
   \( (R_1\cdot) \ [w_1][\bar{a}_1] = [x][y] \)
   we choose \( 0 \leq i_1 \leq n - 1 \) such that \( i_1 + \lambda(w_1) \equiv 0 \).

We obtain the first equality of \( \tilde{\sigma}(I) \) by multiplying the first 
equality of \( \overline{\sigma}(I) \) on the right by \( a^{i_1} \) in the first case 
and on the left by \( a^{j_1} \) in the second case.

We obtain the second equality of \( \tilde{\sigma}(I) \) from the 
second equality of \( \overline{\sigma}(I) \) in the following way: if the first 
(second) factor of the left member of the second 
equality of \( \overline{\sigma}(I) \) is equal to the first (second) factor of 
the right member in the first equality of \( \overline{\sigma}(I) \) then we 
multiply the second equality of \( \overline{\sigma}(I) \) by the left by \( a^{i_1} \) 
and by the right by \( a^{j_1} \) (respectively, by the left by \( a^{i_1} \) 
and by the right by \( a^{j_1} \)) where \( 0 \leq i_2, j_2 \leq n - 1 \) are such 
that the length of the left member of new equality be 
\( \equiv 1 \).

In the same manner we obtain the \( k \text{th} \) equality of 
\( \tilde{\sigma}(I) \) from the \( k \text{th} \) equality of \( \overline{\sigma}(I) \).

**Example.** We apply this procedure to the system 
\( \overline{\sigma}(I) \) considered in the previous example.

Suppose \( n = 5 \), \( \lambda(u_1) = 2 \), \( \lambda(s_1) = 3 \), \( \lambda(s_2) = 3 \), \( \lambda(w_2) = 2 \), \( \lambda(w_3) = 1 \), \( \lambda(w_1) = 4 \), \( \lambda(w_1) = 4 \), \( \lambda(t_1) = 2 \), \( \lambda(w_2) = 2 \), \( \lambda(w_3) = 2 \), \( \lambda(t_3) = 1 \).

The tagged system \( \tilde{\sigma}(I) \) is

\[
\begin{align*}
(R_1L_1) & \ w_1\bar{a}_1 \equiv u_1s_1 \\
(L_1R_2) & \ aa_1s_1 \equiv aw_2t_2 \\
(R_2L_2) & \ aw_2\bar{a}_2 \equiv aw_3s_3a \\
(L_2R_3) & \ a_2w_3s_3a \equiv a_2w_3t_3a \\
(R_3L_2) & \ a_2w_3\bar{a}_3 \equiv a_2w_3s_2a \\
([T_2][\bar{T}_3]) & \ aw_2\bar{s}_2 \equiv aw_3s_3a \\
([\bar{T}_3][\bar{T}_2]) & \ aw_3s_3a \equiv aw_3\bar{a}_2a \\
([\bar{T}_2][\bar{T}_1]) & \ aw_2t_2 \equiv aa_1\bar{s}_1 \\
([T_1][\bar{T}_1]) & \ u_1s_1 \equiv \bar{w}_1\bar{a}_1 \\
([\bar{T}_1][R_1]) & \ \bar{w}_1t_1a^3 \equiv w_1t_1a^3
\end{align*}
\]
Now we prove that $\sigma(I)$ is a system of equalities corresponding to same Malcev sequence $I$. Hence, we must show that the equalities of $\sigma(I)$ are obtained according the table

\[
\begin{array}{c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 \\
\hline
(a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) \\
(a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j}) & (a^k a_0)(s_q a_j^{x j})
\end{array}
\] (2)

Let be $L_q$ any symbol of $I$. We use an inductive argument on $n(L_q) =$ the number of $L$ symbols between $L_q$ and $L_q$. Suppose $n(L_q) = 0$. Then $q = 1$. We have two cases.

**Case 1.** $L_1$ is the first symbol of $I$. Then

\[
\begin{align*}
(L_1 - a_1 a_1 a_j^1 & \equiv x_1 y_1 a_j^1 \\
(-T_1) x_2 y_2 a_j^2 & \equiv a_1 a_1 a_j^2 \\
(T_1 - a_1^2 a_2^1 a_j^2 & \equiv a_1 a_1 a_j^2 \\
(-L_1) a_1^2 a_2^1 a_j^2 & \equiv a_1 a_1 a_j^2
\end{align*}
\]

We have that $\lambda(s_1) + j_1 \equiv 0$, $\lambda(s_1) + j_2 \equiv 0$, $i_1 + \lambda(u_1) + \lambda(s_1) + j_2 \equiv 1$ and $i_1 + \lambda(u_1) + \lambda(s_1) + j_3 \equiv 1$. Hence $i_1 + \lambda(u_1) \equiv 1$ and then $\lambda(s_1) + j_3 \equiv 0$ implies $j_3 = j_1$ and

\[
\begin{array}{c|c|c|c|c|c}
L_1 & L_1 & L_1 & L_1 & L_1 & L_1 \\
\hline
(a_1^1 a_1^1)(s_1^j a_j^1) & (a_1^1 a_1^1)(s_1^j a_j^1) & (a_1^1 a_1^1)(s_1^j a_j^1) & (a_1^1 a_1^1)(s_1^j a_j^1) & (a_1^1 a_1^1)(s_1^j a_j^1)
\end{array}
\] (3)

**Case 2.** $L_q$ is not the first symbol of $I$. Then

\[
\begin{align*}
(L_q - a_1^1 a_2^1 a_j^1 & \equiv a_1^1 a_2^1 a_j^1 \\
(-T_q) a_1^2 a_2^1 a_j^1 & \equiv a_1^2 a_2^1 a_j^1 \\
(T_q - a_1^2 a_2^1 a_j^1 & \equiv a_1^2 a_2^1 a_j^1 \\
(-L_q) a_1^2 a_2^1 a_j^1 & \equiv a_1^2 a_2^1 a_j^1 \\
(T_q - a_1^1 a_2^1 a_j^1 & \equiv a_1^1 a_2^1 a_j^1 \\
(-L_q) a_1^1 a_2^1 a_j^1 & \equiv a_1^1 a_2^1 a_j^1
\end{align*}
\]

We have that $i_1 + \lambda(u_2) + \lambda(s_2) + j_2 \equiv 1$, $i_2 + 1 = i_1 + \lambda(u_2)$, $i_2 + 1 + \lambda(s_2) + j_2 \equiv 1$ and $i_3 + \lambda(u_2) + \lambda(s_2) + j_2 \equiv 1$. Hence $i_3 + \lambda(u_2) = i_2 + 1 = i_1 + \lambda(u_2)$ and thus $i_3 = i_1$, and then

\[
\begin{array}{c|c|c|c|c|c}
L_q & L_q & L_q & L_q & L_q & L_q \\
\hline
(a_1^2 a_2^1)(s_1^j a_j^1) & (a_1^1 a_2^1)(s_1^j a_j^1) & (a_1^1 a_2^1)(s_1^j a_j^1) & (a_1^1 a_2^1)(s_1^j a_j^1) & (a_1^1 a_2^1)(s_1^j a_j^1)
\end{array}
\] (4)

Since $n(L_2) < d$, from

\[
\begin{align*}
L_2 & \equiv (a_2^1 a_2^1)(s_2 a_j^2) \\
(u_2) & \equiv (a_2^1 a_2^1)(s_2 a_j^2) \\
(a_2^1 a_2^1) & \equiv (a_2^1 a_2^1)(s_2 a_j^2)
\end{align*}
\] (5)

it follows that $i_2 = i_3$ and $i_2 = 0$. Now from $i_4 + \lambda(s_1) + j_2 \equiv 1$ it follows $\lambda(s_1) + j_2 \equiv 1$, and from $i_5 + \lambda(u_1) + \lambda(s_1) + j_3 \equiv 1$ we obtain $i_5 + \lambda(u_1) \equiv 1$. Now $i_5 + \lambda(u_1) + \lambda(s_1) + j_3 \equiv 1$ implies $\lambda(s_1) + j_3 \equiv 0$. From the first equality we obtain $\lambda(s_1) + j_1 \equiv 0$. Therefore, $j_5 = j_1$ and we have

\[
\begin{align*}
L_1 & \equiv (a_1^1 a_2^1)(s_1^j a_j^1) \\
(u_1) & \equiv (a_1^1 a_2^1)(s_1^j a_j^1) \\
(a_1^1 a_2^1) & \equiv (a_1^1 a_2^1)(s_1^j a_j^1)
\end{align*}
\] (6)
Case 2. \(L_q\) is not the first symbol of \(I\). Then

\[
\begin{align*}
(-L_q) a^i x y a^k &
\equiv a^i u_q s_q a^k \\
(L_q -) a^i a_q s_q a^k &
\equiv a^i x y a^k \\
(-L_{q+1}) a^i x y a^{k+1} &
\equiv a^i u_{q+1} s_{q+1} a^{k+1} \\
(L_{q+1} -) a^{i+1} a_{q+1} s_{q+1} a^{k+1} &
\equiv a^{i+1} x y a^{k+1} \\
(-L_{q+1}) a^i x y a^{k+1} &
\equiv a^i u_{q+1} s_{q+1} a^{k+1} \\
(L_{q+1} -) a^i a_{q+1} s_{q+1} a^{k+1} &
\equiv a^i x y a^{k+1} \\
(-L_q) a^i x y a^{k+1} &
\equiv a^i u_q s_q a^{k+1} \\
(L_q -) a^i a_q s_q a^{k+1} &
\equiv a^i x y a^{k+1} \\
\end{align*}
\]

\(\) 

We have

\[
\begin{align*}
&i_q + \lambda(u_q) + \lambda(s_q) + j_q \equiv 1 \\
&i_q' + 1 + \lambda(s_q) + j_q \equiv 1 \\
&i_q'' + 1 + \lambda(s_q) + j_q \equiv 1 \\
&i_q'' + 1 + \lambda(s_q) + j_q' \equiv 1 \\
&i_q''' + \lambda(u_q) + \lambda(s_q) + j_q' \equiv 1 \\
&i_q'' + \lambda(u_q) + \lambda(s_q) + j_q' \equiv 1 \\
&i_q'' + \lambda(u_q) + \lambda(s_q) + j_q'' + j_q' \equiv 1 \\
\end{align*}
\]

Since \(n(L_{q+1}) = d - 1\) from

\[
\begin{align*}
\frac{L_{q+1}}{(a^i a_q)(s_q a^{k+1})} \quad &\frac{\bar{L}_{q+1}}{(a^i a_{q+1})(s_{q+1} a^{k+1})} \\
\frac{L_q}{(a^i a_q)(s_q a^{k+1})} \quad &\frac{\bar{L}_q}{(a^i a_q)(s_q a^{k+1})} \\
\end{align*}
\]

it follows that \(i_{q+1} = i_q'' + 1\) and \(i_q'' = i_q'' + 1\).

\[
\begin{align*}
&\text{Now from } i_q'' + 1 + \lambda(s_q) + j_q \equiv 1 \text{ and } i_q'' + 1 + \lambda(s_q) + j_q' \equiv 1, \\
&\text{we get } \lambda(s_q) + j_q \equiv 1, \lambda(s_q) + j_q' \equiv 1. \\
&\text{From } i_q'' + 1 + \lambda(u_q) + \lambda(s_q) + j_q' \equiv 1, \text{ we get } \lambda(s_q) + j_q' \equiv 1. \\
&\text{From } i_q'' + 1 + \lambda(u_q) + \lambda(s_q) + j_q' \equiv 1, \text{ we get } i_q'' + 1 + \lambda(u_q) + \lambda(s_q) + j_q' \equiv 1, \text{ and } i_q + \lambda(u_q) + \lambda(s_q) + j_q \equiv 1, \\
&\text{it follows that } i_q'' + 1 + \lambda(u_q) \equiv 1, \text{ therefore } i_q \equiv i_q'' + 1, \text{ and we have} \\
\end{align*}
\]

\[
\begin{align*}
\frac{L_q}{(a^i a_q)(s_q a^{k+1})} \quad &\frac{\bar{L}_q}{(a^i a_q)(s_q a^{k+1})} \\
\frac{L_q}{(a^i a_q)(s_q a^{k+1})} \quad &\frac{\bar{L}_q}{(a^i a_q)(s_q a^{k+1})} \\
\end{align*}
\]

Similar arguments for \(R\) symbols complete the proof.

Example The corresponding table for \(\tilde{I}(I)\) considered above is

\[
\begin{array}{|c|c|c|c|}
\hline
L_1 & \tilde{L}_1 & R_1 & \bar{R}_1 \\
\hline
(a_{a1}) s_1 & u_{a1} s_1 & w_1 a_1 & w_1 (1 a') \\
(a_{a1}) s_1 & u_{a1} s_1 & w_1 (1 a') & u_{a1} \\
\hline
R_2 & \bar{R}_1 & R_3 & \bar{R}_3 \\
\hline
(a_{a2})_2 s_2 & (a_{a2})_2 s_2 & (a_{a2})_2 a_2 & (a_{a2})_2 (a_2 a) \\
(a_{a2})_2 s_2 & (a_{a2})_2 s_2 & (a_{a2})_2 (a_2 a) & (a_{a2})_2 (a_2 a) \\
\hline
\end{array}
\]

All elements of table 2 are long products. It is easy to see that they have length \(n\) or \(2n - 1\). From the definition of the congruence relation \(p\) it follows that if \(x \equiv y \mod p\) and \(\lambda(x), \lambda(y) \equiv 1\), then \(\alpha(x) = \alpha(y)\), where \(\alpha(x), \alpha(y)\) are the corresponding long products.

It is easy to prove that in terms of \(A\) the system \(\tilde{I}(I)\) is a system of equalities corresponding to the same Malcev sequence \(I\) in which appears now \(n\)-ary symbols.

For example, let be

\[
\begin{align*}
L_k & \equiv (a^i a_k) (s_k a^{k+1}) \\
\bar{L}_k & \equiv (a^i a_k) (s_k a^{k+1}) \\
\end{align*}
\]

Case 1. Suppose

\[
i_k + 1 + \lambda(s_k) + j_k = n \\
i_k' + \lambda(u_k) + \lambda(s_k) + j_k = n
\]

Then

\[
\begin{align*}
&\alpha((a^i a_k) (s_k a^{k+1})) = \alpha(a^i_k, a_k, s_k, a^{k+1}) \\
&\alpha((a^i a_k) (s_k a^{k+1})) = \alpha(a^i_k, u_k, s_k, a^{k+1}) \\
\end{align*}
\]

Now if

\[
i_k + 1 + \lambda(s_k) + j_k = n \\
i_k + 1 + \lambda(s_k) + j_k = n
\]

we have

\[
\begin{align*}
&\alpha((a^i a_k) (s_k a^{k+1})) = \alpha(a^i_k, u_k, s_k, a^{k+1}) \\
&\alpha((a^i a_k) (s_k a^{k+1})) = \alpha(a^i_k, u_k, s_k, a^{k+1}) \\
\end{align*}
\]

and we obtain the table

\[
\begin{align*}
&\alpha(a^i_k, u_k, s_k, a^{k+1}) \\
&\alpha(a^i_k, u_k, s_k, a^{k+1})
\end{align*}
\]

Suppose now that

\[
\begin{align*}
&i_k' + \lambda(u_k) + \lambda(s_k) + j_k' = 2n - 1 \\
i_k + 1 + \lambda(s_k) + j_k' = 2n - 1
\end{align*}
\]

Then

\[
\begin{align*}
&\tilde{s}_k = \tilde{s}_k' \cdot \tilde{s}_k'' \\
&\text{such that} \\
&\alpha((a^i a_k) (s_k a^{k+1})) = \alpha(a^i_k, u_k, s_k', \alpha(s_k', a^{k+1})) \\
&\text{and} \\
&\alpha((a^i a_k) (s_k a^{k+1})) = \alpha(a^i_k, u_k, s_k', \alpha(s_k', a^{k+1})).
\end{align*}
\]

We obtain the table

\[
\begin{align*}
&\alpha(a^i_k, u_k, s_k, a^{k+1}) \\
&\alpha(a^i_k, u_k, s_k, a^{k+1})
\end{align*}
\]
Now we can finish this long proof.

Let $I$ be a Malcev sequence and $\sigma(I)$ the corresponding system of equalities in $S(A)/\rho$ and

$$[x][y] = [u][v]$$

(13)

the closing equality of $\sigma(I)$.

For the system $\tilde{\sigma}(I)$ the closing equality is

$$[a^{k+1}][x][y][a^{k+1}] = [a^{k+1}][u][v][a^{k+1}]$$

(14)

wisch is equivalent to

$$\alpha(a^{k+1}, x, y, a^{k+1}) = \alpha(a^{k+1}, u, v, a^{k+1})$$

(15)

But the last equality is the closing equality for $\tilde{\sigma}(I)$ in terms of $A$. By hypothesis, in $A$ are satisfied all $n$-ary Malcev conditions. Consequently this equality holds. Hence, also (14) holds. $S(A)/\rho$ being a cancellation semigroup, from (14) we get (13). Therefore $S(A)/\rho$ is homomorphic embeddable in a group.

Malcev conditions corresponding to Malcev sequences over the subalphabet $\{L_1, T_1, R_1, T_i, R_i, L_n, T_n, R_n, T_i, R_i | i \in \mathbb{N}\}$ of the alphabet of $n$-ary Malcev symbols $\{L_k, T_k, R_k, T_k, R_k | k = 1, 2, \ldots, n-1; i \in \mathbb{N}\}$ are called unary Malcev conditions.

Now we shall prove the following

**Theorem 3.** If in an $n$-ary semigroup $A$ with lateral identity are satisfied all unary Malcev conditions then $A$ can be homomorphic embedded in an $n$-group.

Proof. Let $a_1^{n-1}$ be a lateral identity. For beginner we prove that $A$ is cancellative.

Suppose that $\alpha(u_1^{n-1}, x) = \alpha(u_1^{n-1}, y)$. Then we have

$$\alpha(u_1^{n-1}, x) = \alpha(u_1^{n-1}, \alpha(a_1^{n-1}, x)) =$$

$$= \alpha(u_1^{n-1}, a_1), a_2^{n-1}, x),$$

and

$$x = \alpha(a_1^{n-1}, x) = \alpha(a_1^{n-1}, a_1), a_2^{n-1}, x).$$

Then for $I = L_1^k T_1^k$ and

$$\alpha(a_1^{n-1}, a_1), a_2^{n-1}, x)$$

$$\alpha(a_1^{n-1}, a_1), a_2^{n-1}, x)$$

we have

$$\alpha(a_1^{n-1}, a_1), a_2^{n-1}, x) = \alpha(a_1^{n-1}, a_1), a_2^{n-1}, y)$$

implies

$$\alpha(a_1^{n-1}, a_1), a_2^{n-1}, y) = \alpha(a_1^{n-1}, a_1), a_2^{n-1}, x)$$

that is

$$\alpha(a_1^{n-1}, x) = \alpha(a_1^{n-1}, y) \Rightarrow x = y.$$
Using Lemma 2 it is easy to prove (by induction) that

**Lemma 3.** In a cancellation $n$-semigroup any circular permutation of a lateral identity is a lateral identity too.

Suppose now that $A$ is a cancellation $n$-semigroup. It is easy to prove that the above endomorphism is in fact an automorphism $xf^{n-1} = a(\alpha_{i-1}^{-1}, y, a_{n-1})$, $xf^{n-1} \cdot a = a \cdot x, \forall x \in A$

where $a = a \cdot x$. We assume that all unary Malcev conditions are satisfied in $A$.

It is easy to prove that the above endomorphism is in fact an automorphism $xf^{n-1} = a(\alpha_{i-1}^{-1}, y, a_{n-1})$.

Let now $A$ be a free group over semigroup $A$, it follows that the semigroup $(A, \cdot)$ is homomorphic embeddable in a group.

We assume that all unary Malcev conditions are satisfied in $A$. Then $A$ is a cancellation $n$-semigroup (see the first part of the proof of Theorem 3). Since the table

<table>
<thead>
<tr>
<th>$L_k$</th>
<th>$\bar{L}_k$</th>
<th>$R_k$</th>
<th>$\bar{R}_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_ky_k$</td>
<td>$w_ky_k$</td>
<td>$w_ky_k$</td>
<td>$w_ky_k$</td>
</tr>
<tr>
<td>$y_kx_k$</td>
<td>$x_ky_k$</td>
<td>$w_ky_k$</td>
<td>$w_kx_k$</td>
</tr>
</tbody>
</table>

is equivalent to the table

| $\alpha(x_k, y_k^{-1}, \ldots, y_{n-1})$ | $\alpha(y_k, x_{n-1}^{-1}, \ldots, x_1)$ | $\alpha(w_k, x_{n-1}^{-1}, \ldots, x_1)$ | $\alpha(w_k, y_k^{-1}, \ldots, y_{n-1})$ | $\alpha(w_k, x_{n-1}^{-1}, y_{n-1})$ |

in $A$, it follows that the semigroup $(A, \cdot)$ is homomorphic embeddable in a group.

Let now $(\mu, (G, \cdot))$ be a free group over semigroup $(A, \cdot)$ (see [3],[4]). Then $\mu$ is a monomorphism. We extend the automorphism $f : (A, \cdot) \to (A, \cdot)$

to an automorphism $\bar{f} : (G, \cdot) \to (G, \cdot)$

such that $\mu \bar{f} = f \mu$.

Now we have $\alpha(x_k^n) \mu \bar{f} = \alpha(x_k^n) f \mu = (x_1 \cdot x_2 \cdot \ldots \cdot x_n f^{n-1} \cdot a) f \mu = x_1 f \mu \cdot x_2 f \mu \cdot \ldots \cdot x_n f \mu \cdot a f \mu = x_1 \mu \bar{f} \cdot x_2 \mu \bar{f} \cdot \ldots \cdot x_n \mu \bar{f} \cdot a \mu$.

Let be $\beta : G^n \to G$ defined by

$$\beta(y_k^n) = y_1 \cdot y_2 \bar{f} \cdot \ldots \cdot y_n f^{n-1} \cdot a \mu.$$

Then

$$\alpha(x_k^n) \mu \bar{f} = \beta(x_1 \mu \bar{f}, \ldots, x_n \mu \bar{f}).$$

Finally we prove that $(G, \beta)$ is an $n$-group. For all $x \in A$ we have

$$x \mu f^{n-1} \cdot a \mu = x f^{n-1} \cdot a \mu = (x f^{n-1} a) \mu = (a x) \mu = a \mu \cdot x \mu,$$

therefore

$$x \mu f^{n-1} = a \mu \cdot x \mu \cdot (a \mu)^{-1}.$$

The set $A \mu$ being a generating subset of the group $(G, \cdot)$ for any $y \in G$

$$y = (x_1 \mu)^{e_1} \cdot (x_2 \mu)^{e_2} \cdot \ldots \cdot (x_k \mu)^{e_k},$$

$x_i \in A$, $e_i = \pm 1$ for all $i = 1, 2, \ldots, k$. Then

$$y f^{n-1} = (x_1 \mu f^{n-1})^{e_1} \cdot (x_2 \mu f^{n-1})^{e_2} \cdot \ldots \cdot (x_k \mu f^{n-1})^{e_k} = (a \mu \cdot x_1 \mu \cdot (a \mu)^{-1})^{e_1} \cdot (x_2 \mu)^{e_2} \cdot \ldots \cdot (x_k \mu)^{e_k} \cdot (a \mu)^{-1} = a \mu \cdot y \cdot (a \mu)^{-1}$. From [6] it follows that $(G, \beta)$ is an $n$-group.

In conclusion

$$\mu \bar{f} : (A, \alpha) \to (G, \beta)$$

is a homomorphic embedding.

**References**


