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GEOMETRICAL INEQUALITIES IN ACUTE TRIANGLE INVOLVING THE MEDIANS V

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ABSTRACT

The purpose of this paper is to demonstrate a new open question that generalizes previous open questions formulated by researchers in the field of geometrical inequalities. In this sense we have proved that in every acute triangle ABC from $a < b < c$ does not result $a^{-2n} + m_a^{-2n} < b^{-2n} + m_b^{-2n} < c^{-2n} + m_c^{-2n}$, where $n \in \mathbb{N}^*$. For the demonstration we have deduced two theorems that allowed the formulation of the main conclusion.

Keywords: geometrical inequalities, acute triangle, medians, open question, bisectrices, altitudes

1 Introduction

Let us consider the acute triangle ABC with sides $a = BC, b = AC$ and $c = AB$. In [1] appeared the following open question due to Pál Erdős: "if ABC is an acute triangle such that $a < b < c$ then $a + l_a < b + l_b < c + l_c$", where l_a, l_b, l_c means the length of the interior bisectrices corresponding to the sides BC, AC and AB, respectively.

In [2] Mihály Bencze proposed the following open question, which is a generalization of the problem of Pál Erdős: "determine all points $M \in Int(ABC)$, for which in case of $BC < CA < AB$ we have $CB + AA' < CA + BB' < AB + CC'$, where A',B',C' is the intersection of AM, BM, CM with sides BC,CA,AB". Here with $Int(ABC)$ we denote the interior points of the triangle ABC.

If we try for "usual" acute triangles ABC, we can verify the validity of the Erdős inequality. But in [3] we realized to obtain an acute triangle for which the Erdős inequality is false: let ABC be such that $c = 10 + \epsilon, b = 10$ and $a = 1$, where $\epsilon > 0$ is a "very small" positive quantity. Using the trigonometrical way combining with some elementary properties from algebra and mathematical analysis we showed that for this "extreme" acute triangle from $a < b < c$ results $c + l_c < b + l_b$.

In [4] József Sándor proved some new geometrical inequalities using in the open question of Pál Erdős instead of bisectrices altitudes and medians.

In [5] Károly Dáné and Csaba Ignát studied the open question of Mihály Bencze using the computer.

In connection with Erdős problem we formulated the following open question: "if ABC is an acute triangle such that $a < b < c$ then $a^2 + l_a^2 < b^2 + l_b^2 < c^2 + l_c^2$". In [6] we proved the validity of this statement.

At the same time we formulated another new open question: "if ABC is an acute triangle such that $a < b < c$ then $a^4 + l_a^4 < b^4 + l_b^4 < c^4 + l_c^4$". In [7] we realized to find two acute triangles ABC such that in the first triangle from $a < b < c$ ($b = a + \epsilon, c = a + 2\epsilon$ with $\epsilon > 0$ a small positive quantity) we obtained $a^4 + l_a^4 < b^4 + l_b^4$, but in the second triangle from $a < b < c$ ($b = a + \epsilon, c = a\sqrt{2}$ with $\epsilon > 0$ a small positive quantity) we deduced $a^4 + l_a^4 > b^4 + l_b^4$. So the answer to our question is negative.

Next we denote by h_a, h_b and h_c the length of the altitudes of the triangle ABC, which correspond to the sides BC,CA and AB, respectively. Then we can formulate the following more general similar question, replacing in the open question of Pál Erdős the bisectrices with altitudes: "if ABC is an acute triangle such that $a < b < c$ then $a^\alpha + h_a^\alpha < b^\alpha + h_b^\alpha < c^\alpha + h_c^\alpha$, where $\alpha \in \mathbb{R}$ is a real number". In [4] or [8] there is showed, that this problem is true for all $\alpha \in \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Next we denote by m_a, m_b and m_c the length of the medians, which correspond to the sides BC, CA and AB, respectively. Then we can formulate the following more general similar question, replacing in the open question of Pál Erdős the bisectrices with medians: "if ABC is an acute triangle from $a < b < c$ results $a^\alpha + m_a^\alpha < b^\alpha + m_b^\alpha < c^\alpha + m_c^\alpha$, where $\alpha \in \mathbb{R}$".

We mention, that this problem is not obvious, because $a < b \Leftrightarrow a^2 < b^2 \Leftrightarrow 2(a^2+c^2) - b^2 < 2(b^2+c^2) - a^2 \Leftrightarrow \sqrt{\frac{2(a^2+c^2)-b^2}{4}} < \sqrt{\frac{2(b^2+c^2)-a^2}{4}} \Leftrightarrow m_b < m_a$.

Similarly $b < c \Leftrightarrow m_b > m_c$.

If $\alpha = 0$, then is immediately that our problem is false.

The above proposed problem is solved in [8] for $\alpha \in \{1, 2, 4\}$ and we obtained that for $\alpha = 1$ it is false and for $\alpha = 2$ and $\alpha = 4$ it is true. In [9] we showed for $\alpha = 8$, that our question is false, and in [10] we proved that for $\alpha = -2$ our problem is false, too. In [11] we obtained for $\alpha = 2n, n \in \mathbb{N}, n \geq 3$, even natural numbers that our statement is false. In [12] we showed for $\alpha = 2n+1, n \in \mathbb{N}, n \geq 2$ odd natural numbers that our affirmation is false.

2 Main part

The purpose of this paper is to study this open question for $\alpha \in \mathbb{Z} - \mathbb{N}$ even, negative integer numbers. Let $\alpha = -2n, n \in \mathbb{N}^* = \mathbb{N} - \{0\}$, be an even, negative integer number.

Theorem 1. *There exists acute triangle ABC with $a < b < c$, such that for this triangle we have $a^{-2n} + m_a^{-2n} < b^{-2n} + m_b^{-2n}$ for every $n \in \mathbb{N}^*$.*

Proof. We have the following sequence of equivalent inequalities: $a^{-2n} + m_a^{-2n} < b^{-2n} + m_b^{-2n} \Leftrightarrow a^{-2n} - b^{-2n} < m_b^{-2n} - m_a^{-2n} \Leftrightarrow \frac{1}{a^{2n}} - \frac{1}{b^{2n}} < \frac{1}{m_b^{2n}} - \frac{1}{m_a^{2n}} \Leftrightarrow \frac{b^{2n}-a^{2n}}{a^{2n}\cdot b^{2n}} < \frac{m_a^{2n}-m_b^{2n}}{m_a^{2n}\cdot m_b^{2n}} \Leftrightarrow \frac{b^2-a^2}{a^{2n}\cdot b^{2n}} \cdot [(b^2)^{n-1} + (b^2)^{n-2}\cdot a^2 + \cdots + b^2 \cdot (a^2)^{n-2} + (a^2)^{n-1}] < \frac{m_a^2 - m_b^2}{m_a^{2n}\cdot m_b^{2n}} \cdot [(m_a^2)^{n-1} + (m_a^2)^{n-2}\cdot m_b^2 + \cdots + (m_a^2)\cdot (m_b^2)^{n-2} + (m_b^2)^{n-1}]$.

But $m_a^2 = \frac{2(b^2+c^2)-a^2}{4}$ and $m_b^2 = \frac{2(a^2+c^2)-b^2}{4}$, so $m_a^2 - m_b^2 = \frac{3\cdot(b^2-a^2)}{4} > 0$. This means, that $a^{-2n} + m_a^{-2n} < b^{-2n} + m_b^{-2n} \Leftrightarrow \frac{1}{a^{2n}\cdot b^{2n}} \cdot [(b^2)^{n-1} + (b^2)^{n-2} \cdot a^2 + \cdots + b^2 \cdot (a^2)^{n-2} + (a^2)^{n-1}] < \frac{3}{4\cdot m_a^{2n}\cdot m_b^{2n}} \cdot [(m_a^2)^{n-1} + (m_a^2)^{n-2}\cdot m_b^2 + \cdots + (m_a^2)\cdot (m_b^2)^{n-2} + (m_b^2)^{n-1}]$.

The first time we choose the acute triangle ABC, such that $b = a + \epsilon$ and $c = a + 2\epsilon$, where $\epsilon > 0$ is a small positive quantity. Then

$$\lim_{\epsilon\to 0} m_a^2 = \lim_{\epsilon\to 0} \frac{2(b^2+c^2)-a^2}{4} = \lim_{\epsilon\to 0} \frac{2[(a+\epsilon)^2 + (a+2\epsilon)^2] - a^2}{4} = \frac{3a^2}{4},$$

$$\lim_{\epsilon\to 0} m_b^2 = \lim_{\epsilon\to 0} \frac{2(a^2+c^2)-b^2}{4} = \lim_{\epsilon\to 0} \frac{2[a^2 + (a+2\epsilon)^2] - (a+\epsilon)^2}{4} = \frac{3a^2}{4},$$

$$\lim_{\epsilon\to 0} b = \lim_{\epsilon\to 0} (a+\epsilon) = a.$$

This means, that $\lim_{\epsilon\to 0} \frac{1}{a^{2n}\cdot b^{2n}} \cdot [(b^2)^{n-1} + (b^2)^{n-2}\cdot a^2 + \cdots + b^2\cdot (a^2)^{n-2} + (a^2)^{n-1}] \leq \lim_{\epsilon\to 0} \frac{3}{4\cdot m_a^{2n}\cdot m_b^{2n}} \cdot [(m_a^2)^{n-1} + (m_a^2)^{n-2}\cdot m_b^2 + \cdots + (m_a^2)\cdot (m_b^2)^{n-2} + (m_b^2)^{n-1}]$, i.e. $\frac{1}{a^{2n}\cdot a^{2n}} \cdot (a^2)^{n-1}\cdot n \leq \frac{3}{4\cdot (\frac{3}{4}a^2)^n \cdot (\frac{3}{4}a^2)^n} \cdot (\frac{3}{4}a^2)^n \cdot n$, i.e. $1 \leq \frac{3}{4\cdot(\frac{3}{4})^n\cdot(\frac{3}{4})^n} \cdot (\frac{3}{4})^{n-1}$, so $(\frac{3}{4})^n \leq 1$.

We can see immediately that the inequality $(\frac{3}{4})^n < 1$ is true for all $n \in \mathbb{N}^* = \mathbb{N} - \{0\}$. Using the definition of the limit we can conclude, that there exists a small positive real number $\epsilon_0 > 0$ such that for the triangle $A_0B_0C_0$ with $B_0C_0 = a, A_0C_0 = b = a + \epsilon_0$ and $A_0B_0 = c = a + 2\epsilon_0$ we obtain $a^{-2n} + m_a^{-2n} < b^{-2n} + m_b^{-2n}$. □

Theorem 2. *There exists acute triangle ABC with $a < b < c$, such that for this triangle we have $a^{-2n} + m_a^{-2n} > b^{-2n} + m_b^{-2n}$ for every $n \in \mathbb{N}^*$.*

Proof. Using the above presented sequence of ideas we get similarly, that $a^{-2n} + m_a^{-2n} > b^{-2n} + m_b^{-2n} \Leftrightarrow \frac{1}{a^{2n}\cdot b^{2n}} \cdot [(b^2)^{n-1} + (b^2)^{n-2}\cdot a^2 + \cdots + b^2 \cdot (a^2)^{n-2} + (a^2)^{n-1}] > \frac{3}{4\cdot m_a^{2n}\cdot m_b^{2n}} \cdot [(m_a^2)^{n-1} + (m_a^2)^{n-2}\cdot m_b^2 + \cdots + (m_a^2)\cdot (m_b^2)^{n-2} + (m_b^2)^{n-1}]$.

The second time we choose the acute triangle ABC, such that $b = a + \epsilon$ and $c = a\sqrt{2}$, where $\epsilon > 0$ is a small positive quantity. Then

$$\lim_{\epsilon\to 0} m_a^2 = \lim_{\epsilon\to 0} \frac{2(b^2+c^2)-a^2}{4} = \lim_{\epsilon\to 0} \frac{2[(a+\epsilon)^2 + 2a^2] - a^2}{4} = \frac{5a^2}{4},$$

$$\lim_{\epsilon\to 0} m_b^2 = \lim_{\epsilon\to 0} \frac{2(a^2+c^2)-b^2}{4} = \lim_{\epsilon\to 0} \frac{2[a^2 + 2a^2] - (a+\epsilon)^2}{4} = \frac{5a^2}{4}, \text{ and}$$

$$\lim_{\epsilon\to 0} b = \lim_{\epsilon\to 0} (a+\epsilon) = a.$$

This means, that $\lim_{\epsilon\to 0} \frac{1}{a^{2n}\cdot b^{2n}} \cdot [(b^2)^{n-1} + (b^2)^{n-2}\cdot a^2 + \cdots + b^2\cdot (a^2)^{n-2} + (a^2)^{n-1}] \geq \lim_{\epsilon\to 0} \frac{3}{4\cdot m_a^{2n}\cdot m_b^{2n}} \cdot [(m_a^2)^{n-1} + (m_a^2)^{n-2}\cdot m_b^2 + \cdots + (m_a^2)\cdot (m_b^2)^{n-2} + (m_b^2)^{n-1}]$, i.e. $\frac{1}{a^{2n}\cdot a^{2n}} \cdot (a^2)^{n-1}\cdot n \geq \frac{3}{4\cdot (\frac{5}{4}a^2)^n \cdot (\frac{5}{4}a^2)^n} \cdot (\frac{5}{4}a^2)^n \cdot n$, i.e. $1 \geq \frac{3}{4\cdot(\frac{5}{4})^n\cdot(\frac{5}{4})^n} \cdot (\frac{5}{4})^{n-1}$, so $(\frac{5}{4})^{n+1} \geq \frac{3}{4}$.

But we can see immediately that the inequality $(\frac{5}{4})^{n+1} > \frac{3}{4}$ is true for all $n \in \mathbb{N}^*$. Using the definition of the limit we can conclude, that there exists a small positive real number $\epsilon_1 > 0$ such that for the triangle $A_1B_1C_1$ with $B_1C_1 = a, A_1C_1 = b = a+\epsilon_1$ and $A_1B_1 = c = a\sqrt{2}$ we obtain $a^{-2n} + m_a^{-2n} > b^{-2n} + m_b^{-2n}$. □

3 Discussion and conclusion

In an acute triangle, Pál Erdős has formulated an open question regarding the length of the interior bisectrices. Later it was generalized by Mihály Bencze by determining all points from the intersection of medians. Then, József Sándor proved some new geometrical inequalities using the open question of Pál Erdős instead of bisectrices altitudes and medians. Based on these researches, we have formulated a new open question, a new geometrical inequality regarding medians in acute triangles, that generalizes some other results of the author. We put the question, what happen in the case of $\alpha = -2n, n \in \mathbb{N}^*$, even negative integer. We have formulated two theorems regarding this problem. From these two inequalities we have deduced a main conclusion showing that our proposed open question is not true. Consequently for $\alpha = -2n, n \in \mathbb{N}^*$ our open question is false.

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