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# POSITIVE SOLUTION FOR M-POINT SIXTH-ORDER BOUNDARY VALUE PROBLEM WITH VARIABLE PARAMETER

**B. Kovács**[1]
**Mohammed Guedda**[2]
[1] *Institute of Mathematics University of Miskolc 3515 Egyetemváros, Hungary*
mathmn@uni-miskolcs.hu
[2] *Faculté de Mathématiques et d'Informatique, Université de Picardie Jules Verne 33*,
*rue Saint-Leu F-80039 Amiens, France*
[2]mohamed.guedda@u-picadrie.fr

## ABSTRACT

*This paper investigates the existence of positive solutions for a sixth-order m-point boundary value problem with three variable parameters. Many problems in the theory of elastic stability can be handled by the method of multi-point problems. By using the fixed point theorem and operator spectral theorem, we give a new existence result.*

**Keywords**: positive solutions, variable parameters, fixed point theorem, operator spectral theorem

## 1 Introduction

Boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics and the existence and multiplicity of positive solutions for such problems has become an important area of investigation in recent years; we refer the reader to [1-15] and the references therein. For example, the deformations of an elastic beam in the equilibrium state can be described as a boundary value problem of some fourth-order differential equations.

Multipoint boundary value problems for ordinary differential equations arise in a variety of areas of applied mathematics and physics. For examples, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem; also many problems in the theory of elastic stability can be handled by the method of multi-point problems. In 2006 Ma [3] studied the existence of positive solutions for the following m-point BVP of fourth order

$$u^{(4)}(t) + \beta u^{(2)}(t) - \alpha u(t) = f(t, u(t)), \quad 0 < t < 1$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$$

$$u^{(2)}(0) = \sum_{i=1}^{m-2} a_i u^{(2)}(\xi_i), \quad u^{(2)}(1) = \sum_{i=1}^{m-2} b_i u^{(2)}(\xi_i)$$

where $\alpha, \beta \in R$, $\xi_i \in (0,1)$, $a_i, b_i \in [0, \infty)$ for $i \in \{1, 2, ..., m-2\}$ are given constants satisfying some suitable conditions.

Recently, Zhang and Wei [4] established the existence result of positive solution for the fourth-order boundary value problem with variable parameters as follows:

$$u^{(4)} + B(t)u^{(2)} - A(t)u(t) = f(t, u(t)), \quad 0 < t < 1$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$$

$$u^{(2)}(0) = \sum_{i=1}^{m-2} a_i u^{(2)}(\xi_i), \quad u^{(2)}(1) = \sum_{i=1}^{m-2} b_i u^{(2)}(\xi_i)$$

It is well known that the deformation of the equilibrium state, an elastic circular ring segment with its two ends simply supported can be described by a boundary value problem for a sixth-order ordinary differential equation:

$$u^{(6)}+2u^{(4)}+u^{(2)}=f(t,u), \quad 0<t<1$$
$$u(0)=u(1)=u^{(2)}(0)=u^{(2)}(1)$$
$$=u^{(4)}(0)=u^{(4)}(1)=0,$$

However, there are only a handful of articles on this topic. See, for example [5-7].

In this paper we shall discuss the existence of positive solutions for the sixth-order boundary value problem

$$-u^{(6)}+A(t)u^{(4)}+B(t)u^{(2)}+C(t)u=f(t,u,u^{(2)}) \tag{1}$$

$$u^{(2i-2)}(0)=\sum_{i=1}^{m-2}a_iu^{(2i-2)}(\xi_i),\ i=1,2,3.$$
$$u^{(2i-2)}(1)=\sum_{i=1}^{m-2}b_iu^{(2i-2)}(\xi_i),\ i=1,2,3. \tag{2}$$

where $A(t),B(t),C(t)\in C[0,1]$. Our results will generalize those established in [3, 4].

For this, we shall assume the following conditions throughout:

(H1) $f:[0,1]\times[0,\infty)\times(-\infty,0]\longrightarrow[0,\infty)$ is continuous.

(H2) $a=\sup_{t\in[0,1]}A(t)>-\pi^2,\ a,b,c\in R,$
$b=\inf_{t\in[0,1]}B(t)>0,$
$c=\sup_{t\in[0,1]}C(t)<0,$

$$\pi^6+a\pi^4-b\pi^2+c>0.$$

Assumption (H2) involves a three-parameter nonresonance condition.

We will apply the cone fixed point theory, combining with the operator spectra theorem to establish the existence of positive solutions of boundary value problem (1-2). The paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we obtain an existence result for the boundary value problem (1-2).

## 2 Preliminaries

Let $Y=C[0,1]$, $Y_+=\{u\in Y: u(t)\geq 0,\ t\in[0,1]\}$. It is well known that $Y$ is a Banach space equipped with the norm $\|u\|_0=\sup_{t\in[0,1]}|u(t)|$.

Set

$$X=\Big\{u\in C^4[0,1]: u^{(2i-2)}(0)= =\sum_{i=1}^{m-2}a_iu^{(2i-2)}(\xi_i),u^{(2i-2)}(1)= =\sum_{i=1}^{m-2}b_iu^{(2i-2)}(\xi_i),\ i=1,2.\Big\}$$

For given $\chi\geq 0$ and $\nu\geq 0$, we denote the norm $\|u\|_{\chi,\nu}$ by $\|u\|_{\chi,\nu}=\sup_{t\in[0,1]}\{|u^{(4)}(t)|+\chi|u^{(2)}(t)|+\ \nu|u(t)|\}$, $u\in X$. We also need the space $X$ equipped with the norm $\|\cdot\|_2=\max\left\{\|u\|_0,\|u^{(2)}\|_0,\|u^{(4)}\|_0\right\}$. In this Section, we will show that $X$ is complete with both the norms $\|\cdot\|_{\chi,\nu}$ and $\|\cdot\|_2$.

Let

$$E=\{C^2[0,1]: u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i),\ u(1) =\sum_{i=1}^{m-2}b_iu(\xi_i),\ u^{(2)}(0) =\sum_{i=1}^{m-2}a_iu^{(2)}(\xi_i),\ u^{(2)}(1) =\sum_{i=1}^{m-2}b_iu^{(2)}(\xi_i)\}.$$

Then $E$ is a Banach space with a norm by

$$\|u\|=\max_{t\in[0,1]}|u^{(2)}(t)|,\ \ \forall u\in E.$$

For $h\in Y$, consider the following linear boundary value problem:

$$-u^{(6)}+au^{(4)}+bu^{(2)}+cu=h(t), \quad 0<t<1 \tag{3}$$

$$u(0)=u(1)=u^{(2)}(0)=u^{(2)}(1)=u^{(4)}(0) =u^{(4)}(1)=0, \tag{4}$$

where $a,b,c$ satisfy the assumption

$$\pi^6+a\pi^4-b\pi^2+c>0 \tag{5}$$

and let $\Gamma=\pi^6+a\pi^4-b\pi^2+c$. The inequality (5) follows immediately from the fact that $\Gamma=\pi^6+a\pi^4-b\pi^2+c$ is the first eigenvalue of the problem $-u^{(6)}+au^{(4)}+bu^{(2)}+cu=\lambda u$, $u(0)=u(1)=u^{(2)}(0)=u^{(2)}(1)=u^{(4)}(0)=u^{(4)}(1)=0$ and $\varphi_1(t)=\sin\pi t$ is the first eigenfunction, i.e. $\Gamma>0$.

Let $P(\lambda)=\lambda^2+\beta\lambda-\alpha$ where $\beta<2\pi^2,\alpha\geq 0$. It is easy to see that equation $P(\lambda)=0$ has two real roots $\lambda_1,\lambda_2=\frac{-\beta\pm\sqrt{\beta^2+4\alpha}}{2}$, with $\lambda_1\geq 0\geq\lambda_2>-\pi^2$. Let $\lambda_3$ be a number such that $0\leq\lambda_3<-\lambda_2$. In this case, (3) satisfies the following decomposition form:

$$-u^{(6)}+au^{(4)}+bu^{(2)}+cu=\left(-\frac{d^2}{dt^2}+\lambda_1\right) \left(-\frac{d^2}{dt^2}+\lambda_2\right)\left(-\frac{d^2}{dt^2}+\lambda_3\right)u, \quad 0<t<1. \tag{6}$$

It is obvious that $a=\lambda_1+\lambda_2+\lambda_3>-\pi^2, b=-\lambda_1\lambda_2-\lambda_2\lambda_3-\lambda_1\lambda_3>0, c=\lambda_1\lambda_2\lambda_3<0.$

**Lemma 1.** [3]. *Assume that (H2) holds. Then there exists unique $\varphi_i,\psi_i, i=1,2,3$ satisfying*

$\left\{\begin{array}{c} -\varphi_i^{(2)} + \lambda_i \varphi_i = 0, \\ \varphi_i(0) = 0, \ \ \varphi_i(1) = 1; \end{array}\right\}$
$\left\{\begin{array}{c} -\psi_i^{(2)} + \lambda_i \psi_i = 0, \\ \psi_i(0) = 1, \ \ \psi_i(1) = 0; \end{array}\right\}$ *respectively. Moreover, $\varphi_i$ and $\psi_i$ are positive on* $[0,1]$.

*For $i = 1,2,3$ set $\rho_i = \varphi_i'(0)$,*

$$G_i(t,s) = \frac{1}{\rho_i}\left\{\begin{array}{l} \varphi_i(t)\psi_i(s), \ 0 \le t \le s \le 1, \\ \varphi_i(s)\psi_i(t), \ 0 \le s \le t \le 1. \end{array}\right\} \quad (7)$$

*Then $G_i(t,s)$, $(i = 1,2,3)$ are the Green's function of the linear boundary value problem*

$$-u^{(2)} + \lambda_i u = 0, \quad u(0) = u(1) = 0.$$

We have the following several lemmas, which will be used in the sequence:

**Lemma 2.** [3]. *Let $\omega_i = \sqrt{|\lambda_i|}$, then $G_i(t,s)(i = 1,2,3)$ can be expressed by*

*(i) when $\lambda_i > 0$,*

$$G_i(t,s) = \left\{\begin{array}{l} \frac{\sinh \omega_i t \sinh \omega_i (1-s)}{\omega_i \sinh \omega_i}, \ 0 \le t \le s \le 1 \\ \frac{\sinh \omega_i s \sinh \omega_i (1-t)}{\omega_i \sinh \omega_i}, \ 0 \le s \le t \le 1 \end{array}\right\}$$

*(ii) when $\lambda_i = 0$,*

$$G_i(t,s) = \left\{\begin{array}{l} t(1-s), \ 0 \le t \le s \le 1 \\ s(1-t), \ 0 \le s \le t \le 1 \end{array}\right\}$$

*(iii) when $-\pi^2 < \lambda_i < 0$,*

$$G_i(t,s) = \left\{\begin{array}{l} \frac{\sin \omega_i t \sin \omega_i (1-s)}{\omega_i \sin \omega_i}, \ 0 \le t \le s \le 1 \\ \frac{\sin \omega_i s \sin \omega_i (1-t)}{\omega_i \sin \omega_i}, \ 0 \le s \le t \le 1 \end{array}\right\}.$$

**Lemma 3.** *$G_i(t,s), \varphi_i, \psi_i$ $(i = 1,2)$ have the following properties:*

*(i) $G_i(t,s) > 0, \forall t,s \in (0,1)$;*

*(ii) $G_i(t,s) \le C_i G_i(s,s), \forall\, t,s \in [0,1]$;*

*(iii) $G_i(t,s) \ge \delta_i G_i(t,t) G_i(s,s), \forall\, t,s \in [0,1]$;*

*(iv) $\delta_i G_i(t,t) \le \varphi_i, (t),\ \psi_i(t) \le C_i, \forall\, t \in [0,1]$ where $C_i = 1, \delta_i = \frac{\omega_i}{\sinh \omega_i}$, if $\lambda_i > 0; C_i = 1, \delta_i = 1$, if $\lambda_i = 0; C_i = \frac{1}{\sin \omega_i}, \delta_i = \omega_i \sin \omega_i$, if $-\pi^2 < \lambda_i < 0$.*

*Denote*

$$G(t,s) = \left\{\begin{array}{l} t(1-s), \ 0 \le t \le s \le 1 \\ s(1-t), \ 0 \le s \le t \le 1 \end{array}\right\},$$

$$\Delta = \left|\begin{array}{cc} \sum_{i=1}^{m-2} a_i \xi_i & \sum_{i=1}^{m-2} a_i(1-\xi_i) - 1 \\ \sum_{i=1}^{m-2} b_i \xi_i - 1 & \sum_{i=1}^{m-2} b_i(1-\xi_i) \end{array}\right|.$$

Applying the similar method to the Lemma 2.2 in [3], we can obtain the following lemma:

**Lemma 4.** [3]. *Suppose that (H2) holds. Assume that (H3) $\Delta < 0$,*

*then for any $g \in C[0,1]$, the problem*

$$-u^{(2)} = g(t), 0 < t < 1$$
$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$$

*has a unique solution*

$$u(t) = \int_0^1 G(t,s) g(s) ds + A_0(g) t + B_0(g)(1-t) \quad (8)$$

*where*

$$A_0(g) = -\frac{1}{\Delta}\left|\begin{array}{cc} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s) g(s) ds & \sum_{i=1}^{m-2} a_i(1-\xi_i) - 1 \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s) g(s) ds & \sum_{i=1}^{m-2} b_i(1-\xi_i) \end{array}\right|$$

$$B_0(g) = -\frac{1}{\Delta}\left|\begin{array}{cc} \sum_{i=1}^{m-2} a_i \xi_i & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s) g(s) d \\ \sum_{i=1}^{m-2} b_i \xi_i - 1 & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s) g(s) ds \end{array}\right|.$$

*We can rewrite (8) the following form:*

$$u(t) = \int_0^1 G(t,s)(-u^{(2)}) ds + A_0(-u^{(2)}) t + B_0(-u^{(2)})(1-t) \quad (9)$$

*and it is easy to see that:*

$$u^{(2)}(t) = \int_0^1 G(t,s)(-u^{(4)}) ds + A_0(-u^{(4)}) t + B_0(-u^{(4)})(1-t), \quad (10)$$

*where $u \in X$.*

**Lemma 5.** *One has that for all $u \in E$, $\|u\|_0 \le \sigma \left\|u^{(2)}\right\|_0$. Moreover, $\forall u \in X$, $\|u\|_0 \le \sigma \left\|u^{(2)}\right\|_0 \le \sigma^2 \left\|u^{(4)}\right\|_0$, where $\sigma = 1 + |A_0(1)| + |B_0(1)|$.*

*Proof.* Using (9) and Lemma 3, we have

$$\begin{aligned} |u(t)| &\le \int_0^1 G(s,s) ds |u^{(2)}(s)| \\ &\quad + |A_0(1)||u^{(2)}(s)| + |B_0(1)||u^{(2)}(s)| \\ &\le (1 + |A_0(1)| + |B_0(1)|)\, t \|u^{(2)} t\|_0 \\ &\le \sigma \|u^{(2)}\|_0, \quad t \in [0,1] \end{aligned}$$

and it follows that $\|u\|_0 \le \sigma \left\|u^{(2)}\right\|_0$. Similarly, one can show that $\left\|u^{(2)}\right\|_0 \le \sigma \left\|u^{(4)}\right\|_0$. □

**Lemma 6.** *Let (H2) and (H3) hold, then $X$ is complete with respect to the norm $\|\cdot\|_{\chi,\nu}$, where the constants $\chi \ge 0$, $\nu \ge 0$, and*

$$(1 + \chi + \nu)^{-1} \|\cdot\|_{\chi,\nu} \le \|\cdot\|_2 \le \sigma^2 \|\cdot\|_{\chi,\nu}, \quad (11)$$

*which means that the norms $\|\cdot\|_2$ and $\|\cdot\|_{\chi,\nu}$ are equivalent.*

*Proof.* It is easy to see that $\|u\|_{\chi,\nu}$ and $\|u\|_2$ are both norms on $X$ by Lemma 5, so we only need to show their completeness.

First we show that the norm $\|\cdot\|_{\chi,\nu}$ is equivalent to the norm $\|u\|_2$. In fact, $\forall u \in X$, $t \in [0,1]$,

$$\begin{aligned}&\left|u^{(4)}(t)\right| + \chi\left|u^{(2)}(t)\right| + \nu\left|u(t)\right| \\ &\le \left\|u^{(4)}\right\|_0 + \chi\left\|u^{(2)}(t)\right\|_0 + \nu\left\|u(t)\right\|_0 \\ &\le (1+\chi+\nu)\|u\|_2 .\end{aligned}$$

Thus $\|u\|_{\chi,\nu} \le (1+\chi+\nu)\|u\|_2$.

Also $\forall u \in X$, $t \in [0,1]$, $\left|u^{(4)}(t)\right| \le \left|u^{(4)}(t)\right| + \chi\left|u^{(2)}(t)\right| + \nu\left|u(t)\right| \le \|u\|_{\chi,\nu}$ and so $\left\|u^{(4)}\right\|_0 \le \|u\|_{\chi,\nu} \le \sigma^2\|u\|_{\chi,\nu}$. By Lemma 5, we have $\left\|u^{(2)}\right\|_0 \le \sigma\left\|u^{(4)}\right\|_0 \le \sigma\|u\|_{\chi,\nu}$ and $\|u\|_0 \le \sigma\left\|u^{(2)}\right\|_0 \le \sigma^2\left\|u^{(4)}\right\|_0 \le \sigma^2\|u\|_{\chi,\nu}$. Hence $\|u\|_2 \le \sigma^2\|u\|_{\chi,\nu}$ then (11) is obtained. Thus $\|u\|_2$ is equivalent to $\|u\|_{\chi,\nu}$.

Let us show that $X$ is complete with respect to the norm $\|u\|_2$. Let $\{u_n\}$ be a Cauchy sequence in $X$, i.e. $\|u_n - u_m\|_0 \to 0$, $\left\|u_n^{(2)} - u_m^{(2)}\right\|_0 \to 0$, $\left\|u_n^{(4)} - u_m^{(4)}\right\|_0 \to 0, (n,m\to\infty)$. So, there exist $u, v, w \in Y$ with $\|u_n - u\|_0 \to 0$, $\left\|u_n^{(2)} - v\right\|_0 \to 0$, $\left\|u_n^{(4)} - w\right\|_0 \to 0, (n\to\infty)$. Since $\{u_n\} \subset X$, from Lemma 4 we have for $\forall\, u \in X$

$$\begin{aligned}u_n(t) = &\int_0^1 G(t,s)(-u_n^{(2)}(s))ds \\ &+ A_0(-u_n^{(2)})t + B_0(-u_n^{(2)})(1-t)\end{aligned} \tag{12}$$

and

$$\begin{aligned}u_n^{(2)}(t) = &\int_0^1 G(t,s)(-u_n^{(4)}(s))ds \\ &+ A_0(-u_n^{(4)})t + B_0(-u_n^{(4)})(1-t).\end{aligned} \tag{13}$$

Taking the limit in (12) and (13),

$$u(t) = -\int_0^1 G(t,s)v(s)ds + A_0(-v)t + B_0(-v)(1-t)$$

$$v(t) = -\int_0^1 G(t,s)w(s)ds + A_0(-w)t + B_0(-w)(1-t)$$

and so $u^{(2)} = v$ and $v^{(2)} = w$.

Thus $u \in X$, we have $\|u_n - u\|_2 \to 0\,(n\to\infty)$, and so $(X, \|\cdot\|_2)$ is complete. Now it follows that $(X, \|\cdot\|_{\chi,\nu})$ is complete from the completeness of $(X, \|\cdot\|_2)$. □

Notation. Set

$$\Delta_j = \begin{vmatrix} \sum_{i=1}^{m-2} a_i\varphi_j(\xi_i) & \sum_{i=1}^{m-2} a_i\psi_j(\xi_i) - 1 \\ \sum_{i=1}^{m-2} b_i\varphi_j(\xi_i) - 1 & \sum_{i=1}^{m-2} b_i\psi_j(\xi_i) \end{vmatrix}, \tag{14}$$

$$A_j(g) = -\frac{1}{\Delta_j}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G_j(\xi_i,s)g(s)ds & \sum_{i=1}^{m-2} a_i\psi_j(\xi_i) - 1 \\ \sum_{i=1}^{m-2} b_i\int_0^1 G_j(\xi_i,s)g(s)ds & \sum_{i=1}^{m-2} b_i\psi_j(\xi_i) \end{vmatrix}, \tag{15}$$

$$B_j(g) = -\frac{1}{\Delta_j}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\varphi_j(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G_j(\xi_i,s)g(s)ds \\ \sum_{i=1}^{m-2} b_i\varphi_j(\xi_i) - 1 & \sum_{i=1}^{m-2} b_i\int_0^1 G_j(\xi_i,s)g(s)ds \end{vmatrix}, \tag{16}$$

where $j = 1,2,3$.

**Remark 1.** *For any $g \in Y$, we have*

$$|A_i(g)| \le |A_i(1)|\,\|g\|_0, \qquad |B_i(g)| \le |B_i(1)|\,\|g\|_0,$$

*where $i = 1,2,3$.*

In the rest of the paper, we make the following assumptions:

(A1) $\sum_{i=1}^{m-2} a_i\psi_j(\xi_i) < 1$, $\sum_{i=1}^{m-2} b_i\varphi_j(\xi_i) < 1$; $j = 1,2,3$.

**Lemma 7.** [3]. *Let (H2), (A1) hold. Assume that*

*(H4) $\Delta_j < 0$, $i = 1,2,3$.*

*Then for any $g \in C[0,1]$, the problem*

$$\begin{aligned}&-u^{(2)} + \lambda_i u = g(t), 0 < t < 1 \\ &u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)\end{aligned}$$

*has a unique solution*

$$\begin{aligned}u(t) = &\int_0^1 G_i(t,s)g(s)ds + A_i(g)\varphi_i(t) \\ &+ B_i(g)\psi_i(t).\end{aligned} \tag{17}$$

*Moreover, if $g \ge 0$, then $u(t) \ge 0$, $t \in [0,1]$.*

*Proof.* The proof follows by routine calculations. Since $\Delta_j < 0$, we have $A_i(g) \ge 0$, $B_i(g) \ge 0$, $i = 1,2,3$.

Define an operator $T_i : Y \to Y$ by

$$\begin{aligned}(T_i g)(t) = &\int_0^1 G_i(t,s)g(s)ds + A_i(g)\varphi_i(t) \\ &+ B_i(g)\psi_i(t), \quad i = 1,2,3.\end{aligned} \tag{18}$$

Using Lemma 1. and Lemma 3. we have

$$\begin{aligned}|(T_i g)(t)| &= \bigg| \int_0^1 G_i(t,s)g(s)ds + A_i(g)\varphi_i(t) \\ &\quad + B_i(g)\psi_i(t)\bigg| \\ &\le C_i \int_0^1 G_i(s,s)ds \, \|g\|_0 \\ &\quad + A_i(1)\|g\|_0 \varphi_i(t) + B_i(1)\|g\|_0 \psi_i(t) \\ &\le \{C_i D_i + A_i(1)E_i + B_i(1)F_i\} \|g\|_0 \\ &= M_i \|g\|_0\end{aligned}$$

where $M_i = C_i D_i + A_i(1)E_i + B_i(1)F_i$, $D_i = \int_0^1 G_i(s,s)ds$, $E_i = \max_{t\in[0,1]} |\varphi_i(t)|$, and $F_i = \max_{t\in[0,1]} |\psi_i(t)|$.

Thus $\|T_i g\|_0 \le M_i \|g\|_0$, and so

$$\|T_i\| \le M_i, \quad i = 1,2,3. \tag{19}$$

□

Notice that

$$\begin{aligned}&-u^{(6)} + au^{(4)} + bu^{(2)} + cu = \Big(-\frac{d^2}{dt^2} + \lambda_1\Big) \\ &\Big(-\frac{d^2}{dt^2} + \lambda_2\Big)\Big(-\frac{d^2}{dt^2} + \lambda_3\Big)u = h(t)\end{aligned} \tag{20}$$

so we can easily get:

**Lemma 8.** *Let (H2), (H3), (H4) and (A1) hold. Then for any $h \in Y$, the problem:*

$$-u^{(6)} + au^{(4)} + bu^{(2)} + cu = h(t), 0 < t < 1 \tag{21}$$

$$\begin{aligned}u^{(2i-2)}(0) &= \sum_{i=1}^{m-2} a_i u^{(2i-2)}(\xi_i),\ i = 1,2,3 \\ u^{(2i-2)}(1) &= \sum_{i=1}^{m-2} b_i u^{(2i-2)}(\xi_i),\ i = 1,2,3\end{aligned} \tag{22}$$

*has a unique solution*

$$\begin{aligned}u(t) &= \int_0^1\int_0^1\int_0^1 G_3(t,v)G_2(v,\tau)G_1(\tau,s)h(s)dsd\tau dv \\ &\quad + \int_0^1\int_0^1 G_3(t,v)G_2(v,\tau)[A_1(h)\varphi_1(\tau) \\ &\quad + B_1(h)\psi_1(\tau)t]d\tau dv \\ &\quad + \int_0^1 G_3(t,v)[A_2(T_1(h))\varphi_2(v) \\ &\quad + B_2(T_1(h))\psi_2(v)]dv + A_3((T_2T_1)(h))\varphi_3(t) \\ &\quad + B_3((T_2T_1)(h))\psi_3(t), t\epsilon[0,1]\end{aligned} \tag{23}$$

*where*

$$\begin{aligned}T_1(h)(t) &= \int_0^1 G_1(t,s)h(s)ds + A_1(h)\varphi_1(t) \\ &\quad + B_1(h)\psi_1(t)\end{aligned} \tag{24}$$

*and*

$$\begin{aligned}(T_2T_1)(h)(t) &= \int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)h(s)ds \\ &\quad + A_1(h)\varphi_1(\tau) + B_1(h)\psi_1(\tau)d\tau \\ &\quad + A_2(T_1(h))\varphi_2(t) \\ &\quad + B_2(T_1(h))\psi_2(t)\end{aligned} \tag{25}$$

*where $G_i, A_i, B_i, i = 1,2,3$ are defined as in (7), (15) and (16). In addition, if $h \ge 0$, then $u(t) \ge 0$, $t \in [0,1]$.*

*Define an operator $T : Y \to Y$ by*

$$\begin{aligned}(Th)(t) &= (T_3T_2T_1)(h)(t) \\ &= \int_0^1\int_0^1\int_0^1 G_3(t,v)G_2(v,\tau) \\ &\quad G_1(\tau,s)h(s)dsd\tau dv \\ &\quad + \int_0^1\int_0^1 G_3(t,v)G_2(v,\tau)[A_1(h)\varphi_1(\tau) \\ &\quad + B_1(h)\psi_1(\tau)]d\tau dv \\ &\quad + \int_0^1 G_3(t,v)[A_2(T_1(h))\varphi_2(v) \\ &\quad + B_2(T_1(h))\psi_2(v)]dv \\ &\quad + A_3((T_2T_1)(h))\varphi_3(t) \\ &\quad + B_3((T_2T_1)(h))\psi_3(t)\end{aligned} \tag{26}$$

*where $T_1(h)(t)$ and $T_2T_1(h)(t)$ are defined by (24) and (25) respectively.*

**Lemma 9.** *Suppose (H2), (H3), (H4) and (A1) hold, then $T : Y \to (X, \|u\|_{\varkappa,\nu})$ is linear completely continuous where $\chi = \lambda_1 + \lambda_3, \nu = \lambda_1\lambda_3$ and $\|T\| \le M_2$.*

*Proof.* The proof of complete continuous is similar to the proof of Lemma 2.8 in [4], so we omit it. Next we will show that $\|T\| \le M_2$. Assume that $h \in Y$ and $u = Th$ is the solution the boundary value problem (21-22). It is clear that the operator $T$ maps $Y$ into $X$. Using (20) it is easy to see that

$$\begin{aligned}-u^{(2)} + \lambda_i u &= \int_0^1\int_0^1 G_j(t,v)G_k(v,\tau)h(\tau)d\tau \\ &\quad + A_k(h)\varphi_k(v) + B_k(h)\psi_k(v)dv \\ &\quad + A_j(T_k(h))\varphi_j(t) + B_j(T_k(h))\psi_j(t),\end{aligned} \tag{27}$$

and

$$\begin{aligned}&u^{(4)} - (\lambda_i + \lambda_j)u^{(2)} + \lambda_i\lambda_j u \\ &= \int_0^1 G_k(t,v)h(v)dv + A_k(h)\varphi_k(t) \\ &\quad + B_k(h)\psi_k(t),\end{aligned} \tag{28}$$

where $i, j, k = 1,2,3$ and $i \ne j \ne k$.

We will now show $\|Th\|_{\chi,\nu} \le M_2 \|h\|_0, \forall h \in Y$, where $\chi = \lambda_1 + \lambda_3 \ge 0, \nu = \lambda_1\lambda_3 \ge 0$. For this,

$\forall h \in Y_+$, let $u = Th$, and by Lemma 3, $u \in X \cap Y_+$. The equality (27) with the assumption $\lambda_2 \le 0$ implies that $u^{(2)} \le 0$. Similarly, the equality (28) with the assumptions $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 \le 0$ implies that $u^{(4)} \ge 0$.

From (28) with $\chi = \lambda_1 + \lambda_3 \ge 0$, $\nu = \lambda_1\lambda_3 \ge 0$ and $u \ge 0$, $u^{(2)} \le 0$, $u^{(4)} \ge 0$ we immediately have

$$\begin{aligned} &|u^{(4)}(t)| + \chi|u^{(2)}(t)| + \nu|u(t)| \\ &= u^{(4)} - (\lambda_1 + \lambda_3)u^{(2)} + \lambda_1\lambda_3 u \\ &= \int_0^1 G_2(t,v)h(v)dv + A_2(h)\varphi_2(t) \\ &+ B_2(h)\psi_2(t). \end{aligned} \tag{29}$$

For any $h \in Y$, let $h = h_1 - h_2, u_1 = Th_1, u_2 = Th_2$, where $h_1, h_2$ are the positive part and negative part of $h$, respectively. Let $u = Th$, then $u = u_1 - u_2$. From the above, we have $u_i \ge 0, u_i^{(2)} \le 0, u_i^{(4)} \ge 0, i = 1, 2$, and the following equality holds:

$$\begin{aligned} &|u_i^{(4)}(t)| + (\lambda_1 + \lambda_3)|u_i^{(2)}(t)| + \lambda_1\lambda_3|u_i(t)| \\ &= \int_0^1 G_2(t,v)h_i(v)dv + A_2(h_i)\varphi_2(t) \\ &+ B_2(h_i)\psi_2(t) = T_2 h_i. \end{aligned} \tag{30}$$

So, by (30), we have

$$\begin{aligned} &|u^{(4)}(t)| + (\lambda_1 + \lambda_3)|u^{(2)}(t)| + \lambda_1\lambda_3|u(t)| \\ &= |u_1^{(4)}(t) - u_2^{(4)}(t)| + (\lambda_1 + \lambda_3)|u_1^{(2)}(t) - u_2^{(2)}(t)| \\ &+ \lambda_1\lambda_3|u_1(t) - u_2(t)| \le (|u_1^{(4)}(t)| \\ &+ (\lambda_1 + \lambda_3)|u_1^{(2)}(t)| + \lambda_1\lambda_3|u_1(t)|) \\ &+ \left(|u_2^{(4)}(t)| + (\lambda_1 + \lambda_3)|u_2^{(2)}(t)| + \lambda_1\lambda_3|u_2(t)|\right) \\ &= T_2 h_1 + T_2 h_2 = T_2|h| \le (C_2 D_2 + A_2(1)E_2 \\ &+ B_2(1)F_2)\, \||h|\|_0 = M_2 \|h\|_0 . \end{aligned}$$

Thus $\|Th\|_{\chi,\nu} \le M_2 \|h\|_0$, and so $\|T\| \le M_2$. □

**Lemma 10.** *Let $f_n : (0,1) \to R$ be a sequence of a continuously differentiable functions. If*

*i) $\lim_{n\to\infty} f_n(x) = f(x)$ on $(0,1)$, and*

*ii) $\lim_{n\to\infty} f_n'(x) = p(x)$, where convergence is uniform on $(0,1)$,*

*then $f(x)$ is continuously differentiable on $(0,1)$, and for all $x \in (0,1)$ we have*

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

*We list the following conditions for convenience:*

*Let $a, b, c \in R$, $a = \lambda_1 + \lambda_2 + \lambda_3 > -\pi^2, b = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 > 0, c = \lambda_1\lambda_2\lambda_3 < 0$ where $\lambda_1 \ge 0 \ge \lambda_2 \ge -\pi^2, 0 \le \lambda_3 < -\lambda_2$ and $\pi^6 + a\pi^4 - b\pi^2 + c > 0$. Let $a = \sup_{t\in[0,1]} A(t)$, $b = \inf_{t\in[0,1]} B(t)$, $c = \sup_{t\in[0,1]} C(t)$. Let $K = \max_{0\le t\le 1}[-A(t) + B(t) - C(t) - (-a + b - c)]$, $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$, $\Gamma_1 = \frac{1-L}{\sigma C_1 C_2 C_3 N_1 N_3}$, $L_1 = M_1 M_2 M_3 K$, $L = K M_2$.*

## 3 Main results

**Theorem 1.** *Assume that $(H_2), (H_3), (H_4)$ and $(A1)$ hold, and $L < 1$, $L_1 < 1$. If*

$$\lim_{|u|+|v|\to 0+} \inf \min_{t\in[0,1]} \frac{f(t,u,v)}{|u| + |v|} > \Gamma$$

*and*

$$\lim_{|v|\to\infty} \sup \max_{t\in[0,1]} \sup_{u\in[0,\infty)} \frac{f(t,u,v)}{|v|} < \Gamma_1$$

*then BVP (1-2) has at least one positive solution.*

*Proof.* Step 1. We consider the existence of positive solution of (1-2) (the function $u \in C^6(0,1) \cap C^4[0,1]$ is a positive solution of (1-2), if $u \ge 0$, $t \in [0,1]$, and $u \ne 0$). Consider the following boundary value problem:

$$\begin{aligned} &-u^{(6)} + au^{(4)} + bu^{(2)} + cu = -(A(t) - a)u^{(4)} \\ &- (B(t) - b)\, u^{(2)} - (C(t) - c)\, u + h(t), \end{aligned} \tag{31}$$

$$\begin{aligned} u^{(2i-2)}(0) &= \sum_{i=1}^{m-2} a_i u^{(2i-2)}(\xi_i),\ i = 1, 2, 3 \\ u^{(2i-2)}(1) &= \sum_{i=1}^{m-2} b_i u^{(2i-2)}(\xi_i),\ i = 1, 2, 3. \end{aligned} \tag{32}$$

For any $u \in X$, let

$$Gu = -(A(t) - a)u^{(4)} - (B(t) - b)u^{(2)} - (C(t) - c)u.$$

Obviously, the operator $G : X \to Y$ is linear. By Lemmas 5 and 6, $\forall u \in X, t \in [0,1]$, we have

$$\begin{aligned} |(Gu)(t)| &\le [-A(t) + B(t) - C(t) \\ &\quad - (-a + b - c)]\|u\|_2 \\ &\le K_1\|u\|_2 \le K_1\sigma^2\|u\|_{\chi,\nu} \end{aligned}$$

where

$$K_1 = \max_{t\in[0,1]}[-A(t) + B(t) - C(t) - (-a + b - c)],$$

$\chi = \lambda_1 + \lambda_3 \ge 0, \nu = \lambda_1\lambda_3 \ge 0$. Hence $\|Gu\|_0 \le K\|u\|_{\chi,\nu}$, where $K = K_1\sigma^2$ and so $\|G\| \le K$. Also $u \in C^4[0,1] \cap C^6(0,1)$ is a solution of (31) iff $u \in X$ satisfies $u = T(Gu + h)$, i.e.

$$u \in X,\ (I - TG)\, u = Th. \tag{33}$$

Let $L = M_2 K$. The operator $I - TG$ maps $X$ into $X$. From $\|T\| \le M_2$ together with $\|G\| \le$

$K$ and condition $M_2K < 1$, and applying operator spectra theorem, we find that $(I - TG)^{-1}$ exists and it is bounded.

Step 2 . Let $H = (I - TG)^{-1}T$. Then (33) is equivalent to $u = Hh$. By the Neumann expansion formula, $H$ can be expressed by

$$\begin{aligned} H &= (I + TG + (TG)^2 + \ldots + (TG)^n + \ldots)T \\ &= T + (TG)T + (TG)^2T + \ldots + (TG)^nT + \ldots \end{aligned} \quad (34)$$

The complete continuity of $T$ with the continuity of $(I - TG)^{-1}$ yields that the operator $H : Y \to X$ is completely continuous.

$\forall\, h \in Y_+$, let $u = Th$, the $u \in X \cap Y_+$, and $u^{(2)} \le 0,\ u^{(4)} \ge 0$. Thus we have

$$\begin{aligned} (Gu)(t) = &-(A(t) - a)u^{(4)} - (B(t) - b)u^{(2)} \\ &- (C(t) - c)u \ge 0,\ t \in [0, 1]. \end{aligned}$$

Hence

$$\forall\, h \in Y_+, (GTh)\,(t) \ge 0, \quad t \in [0,1]\,. \quad (35)$$

and so $(TG)\,(Th)\,(t) = T\,(GTh)\,(t) \ge 0, t \in [0,1]\,.$ By induction it is easy to see

$$\forall\, n \ge 1,\ h \in Y_+, (TG)^n(Th)(t) \ge 0, t \in [0,1]. \quad (36)$$

By (34), we have

$$\begin{aligned} &\forall\, h \in Y_+,\ (Hh)(t) = (Th)(t) + (TG)(Th)(t) \\ &+ (TG)^2(Th)(t) + \ldots + (TG)^n(Th)(t) \\ &+ \ldots \ge (Th)(t),\ t \in [0,1]. \end{aligned} \quad (37)$$

and so $H : Y_+ \to Y_+ \cap X$.

On the other hand, we have

$$\begin{aligned} &\forall\, h \in Y_+,\ (Hh)(t) \\ &\le (Th)(t) + \|(TG)\|(Th)(t) \\ &+ \|(TG)\|^2(Th)(t) + \ldots + \|(TG)\|^n(Th)(t) \\ &+ \ldots \le (1 + L + \ldots + L^n + \ldots)(Th)(t) \\ &= \frac{1}{1-L}(Th)(t). \end{aligned} \quad (38)$$

So the following inequalities hold:

$$(Hh)\,(t) \le \frac{1}{1-L}\,\|(Th)\|_0\,, \quad t \in [0,1]\,. \quad (39)$$

$$\|(Hh)\|_0 \le \frac{1}{1-L}\,\|(Th)\|_0\,. \quad (40)$$

For any $u \in Y_+ \cap C^2\,[0,1]\,,$ define $Fu = f(t, u, u^{(2)})$. By assuming $(H_1)\,,$ we have that $F : Y_+ \cap C^2\,[0,1] \to Y_+$ is continuous. It is easy to see that $u \in C^4\,[0,1] \cap C^6\,(0,1)$ being a positive solution of (1-2) is equivalent to $u \in Y_+ \cap C^2\,[0,1]$ being a nonzero solution equation as follows:

$$u = HFu. \quad (41)$$

Let $Q = HF$. Obviously, $Q : Y_+ \cap C^2\,[0,1] \to Y_+ \cap C^2\,[0,1]$ is completely continuous. We next show that the operator $Q$ has a nonzero fixed point in $Y_+ \cap C^2\,[0,1]\,.$

Step 3. From (31) we also have

$$\left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-\frac{d^2}{dt^2} + \lambda_3\right)V_1 = Gu + h(t) \quad (42)$$

$$\begin{aligned} V_1(0) &= \sum_{i=1}^{m-2} a_i V_1(\xi_i), \\ V_1(1) &= \sum_{i=1}^{m-2} b_i V_1(\xi_i) \\ V_1^{(2)}(0) &= \sum_{i=1}^{m-2} a_i V_1^{(2)}(\xi_i), \\ V_1^{(2)}(1) &= \sum_{i=1}^{m-2} b_i V_1^{(2)}(\xi_i) \end{aligned} \quad (43)$$

where $V_1(t) = (-\frac{d^2}{dt^2} + \lambda_2)u$. It is easy to see that $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$. So the following boundary value problem

$$-u^{(2)}(t) + \lambda_2 u(t) = V_1(t), \quad (44)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \quad (45)$$

can be solved by

$$\begin{aligned} u(t) = (T_2V_1)(t) &= \int_0^1 G_2(\tau, s)V_1(s)ds \\ &+ A_2(V_1)\varphi_2(t) + B_2(V_1)\psi_2(t). \end{aligned} \quad (46)$$

Moreover from (42) using (46) we obtain

$$\left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-\frac{d^2}{dt^2} + \lambda_3\right)V_1 = GT_2V_1 + h(t) \quad (47)$$

$$\begin{aligned} V_1(0) &= \sum_{i=1}^{m-2} a_i V_1(\xi_i), \\ V_1(1) &= \sum_{i=1}^{m-2} b_i V_1(\xi_i) \\ V_1^{(2)}(0) &= \sum_{i=1}^{m-2} a_i V_1^{(2)}(\xi_i), \\ V_1^{(2)}(1) &= \sum_{i=1}^{m-2} b_i V_1^{(2)}(\xi_i) \end{aligned} \quad (48)$$

From eq. (47), we have

$$V_1(t) = T_3T_1(GT_2V_1 + h(t)).$$

On the other hand, $V_1 \in C^2[0,1] \cap C^4(0,1)$ is a solution of (47-48) iff $V_1(t)$ satisfies $V_1 = T_3T_1(GT_2V_1 + h)$, i.e.

$$(I - T_3T_1GT_2)V_1 = T_3T_1h. \tag{49}$$

From $\|T_3T_1\| \leq M_3M_1$, $\|T_2\| \leq M_2$ together with $\|G\| \leq K$ and condition $M_1M_2M_3K < 1$, applying operator spectra theorem, we have that the $(I - T_3T_1GT_2)^{-1}$ exists and it is bounded. Let $L_1 = M_1M_2M_3K$.

Let $H_1 = (I - T_3T_1GT_2)^{-1}T_3T_1$ then (49) is equivalent to $V_1 = H_1h$. By the Neumann expansion formula, $H_1$ can be expressed by

$$\begin{aligned} H_1 &= \big(I + T_3T_1GT_2 + (T_3T_1GT_2)^2 \\ &\quad + \ldots + (T_3T_1GT_2)^n + \ldots\big)\, T_3T_1 = T_3T_1 \\ &\quad + (T_3T_1GT_2)T_3T_1 + (T_3T_1GT_2)^2\, T_3T_1 \\ &\quad + \ldots + (T_3T_1GT_2)^n\, T_3T_1 + \ldots. \end{aligned} \tag{50}$$

The complete continuity of $T_3T_1$ with the continuity of $(I - T_3T_1GT_2)^{-1}$ yields that the operator $H_1 : Y \to C^2[0,1]$ is completely continuous.

By (50), we have $\forall\, h \in Y_+$,

$$\begin{aligned} (H_1h)(t) &= (T_3T_1h)(t) \\ &\quad + ((T_3T_1GT_2)T_3T_1h)(t) \\ &\quad + ((T_3T_1GT_2)^2T_3T_1h)(t) \\ &\quad + \ldots + ((T_3T_1GT_2)^nT_3T_1h)(t) \\ &\quad + \ldots \geq (T_3T_1h)(t), \quad t \in [0,1]. \end{aligned} \tag{51}$$

and so $H_1 : Y_+ \to Y_+ \cap C^2[0,1]$.

On the other hand, we have $\forall\, h \in Y_+$,

$$\begin{aligned} (H_1h)(t) &\leq (Th)(t) \\ &\quad + \|(T_3T_1GT_2)\|(T_3T_1h)(t) \\ &\quad + \|(T_3T_1GT_2)\|^2(T_3T_1h)(t) \\ &\quad + \ldots + \|(T_3T_1GT_2)\|^n(T_3T_1h)(t) \\ &\quad + \ldots \leq (1 + L_1 + \ldots + L_1^n + \ldots) \\ &\qquad (T_3T_1h)(t) \\ &= \frac{1}{1 - L_1}(T_3T_1h)(t). \end{aligned} \tag{52}$$

So the following inequalities hold:

$$(H_1h)(t) \leq \frac{1}{1 - L_1}\, \|(T_3T_1h)\|_0, \tag{53}$$

$$\|(H_1h)\|_0 \leq \frac{1}{1 - L_1}\, \|(T_3T_1h)\|_0. \tag{54}$$

Moreover from (44) using (34) and (50) we obtain

$$u^{(2)}(t) = \lambda_2u(t) - V_1(t) = \lambda_2Hh(t) - H_1h(t) \leq 0,$$

where $\lambda_2 \leq 0$. Let $E(t) = G_3(t,t)$.

Let

$$\begin{aligned} P = \{u \in E : u(t) \geq 0,\ u(t) \\ \geq \Theta E(t)\|u\|_0,\ -u^{(2)}(t) \\ \geq \Theta_2E(t)\|u^{(2)}\|_0,\ t \in [0,1]\}, \end{aligned}$$

where

$$\begin{aligned} \Theta &= \frac{\delta_1\delta_2\delta_3}{C_1C_2C_3N_3}(1 - L), \\ \Theta_2 &= \frac{\delta_1\delta_3\left(-\lambda_2\delta_2\widehat{G}_1 + \widehat{G}_2\right)}{M\left(-\lambda_2C_1C_2C_3N_3 + C_1C_3N_4\right)}, \\ N_3 &= D_2D_3 + D_3\left[A_2(1) + B_2(1)\right] \\ &\quad + \left(A_3(1) + B_3(1)\right)\left(D_2 + A_2(1) + B_2(1)\right), \\ \widehat{G}_1 &= g_{32}g_{21} + g_{32}\left[A_2(G_1) + B_2(G_1)\right] \\ &\quad + \left[A_3(G_2) + B_3(G_2)\right] \\ &\qquad [g_{21} + A_2(G_1) + B_2(G_1)]), \\ \widehat{G}_2 &= g_{31} + A_3(G_1) + B_3(G_1), \\ N_4 &= D_3 + A_3(1) + B_3(1), \\ M &= \max\left\{\frac{1}{1 - L}, \frac{1}{1 - L_1}\right\}. \end{aligned}$$

Step 4. It is easy to see that $P$ is a cone in $E$. Now we show $QP \subset P$. For $\forall u \in P$, let $h_1 = Fu$, then $h_1 \in Y_+$. From (37), $(Qu)(t) = (HFu)(t) \geq (TFu)(t)$, $t \in [0,1]$. From Lemma 3 for all $u \in P$, we have

$$\begin{aligned} &(TFu)(t) \\ &= \int_0^1\int_0^1\int_0^1 G_3(t,v)G_2(v,\tau)G_1(\tau,s)(Fu)(s)ds\,d\tau\,dv \\ &+ \int_0^1\int_0^1 G_3(t,v)G_2(v,\tau)[A_1(Fu)\varphi_1(\tau) \\ &+ B_1(Fu)\psi_1(\tau)]d\tau + \int_0^1 G_3(t,v)[A_2(T_1(Fu))\varphi_2(v) \\ &+ B_2(T_1(Fu))\psi_2(v)]dv + A_3(T_2T_1(Fu))\varphi_3(t) \\ &+ B_3(T_2T_1(Fu))\psi_3(t) \leq C_1C_2C_3\left[\int_0^1 G_3(v,v)dv\right] \\ &\left[\int_0^1 G_2(\tau,\tau)d\tau\right]t\left[\int_0^1 G_1(s,s)(Fu)(s)ds\right] \\ &+ C_1C_2C_3\left[\int_0^1 G_3(v,v)dv\right]\left[\int_0^1 G_2(\tau,\tau)d\tau\right] \\ &\left[A_1(Fu) + B_1(Fu)\right] + C_1C_2C_3\left[\int_0^1 G_3(v,v)dv\right] \\ &[A_2(1) + B_2(1)] \cdot \left[\int_0^1 G_1(s,s)(Fu)(s)ds\right. \\ &\left. + A_1(Fu) + B_1(Fu)\right] + C_1C_2C_3[A_3(1) \end{aligned}$$

$$+ B_3(1)]\left[\int_0^1 G_2(\tau,\tau)d\tau + A_2(1) + B_2(1)t\right]$$
$$\cdot\left[\int_0^1 G_1(s,s)(Fu)(s)ds + A_1(Fu) + B_1(Fu)\right]$$
$$= C_1C_2C_3N_3\left[\int_0^1 G_1(s,s)(Fu)(s)ds + A_1(Fu) + B_1(Fu)\right].$$

where $T_1(h)(t)$ and $T_2T_1(h)(t)$ is defined by (24) and (25) respectively.

Thus

$$\int_0^1 G_1(s,s)(Fu)(s)ds + A_1(Fu) + B_1(Fu) \geq \frac{1}{C_1C_2C_3N_3}\|TFu\|_0. \quad (55)$$

Also from (40) and (55) we have

$$(Qu)(t) \geq (TFu)(t) \geq$$
$$\delta_1\delta_2\delta_3G_3(t,t)\left[\int_0^1 G_3(v,v)G_2(v,v)dv\right]$$
$$\cdot\left[\int_0^1 G_2(\tau,\tau)G_1(\tau,\tau)d\tau\right]\left[\int_0^1 G_1(s,s)(Fu)(s)ds\right]$$
$$+ \delta_1\delta_2\delta_3G_3(t,t)\left[\int_0^1 G_3(v,v)G_2(v,v)dv\right]$$
$$\cdot\left[\int_0^1 G_2(\tau,\tau)G_1(\tau,\tau)d\tau\right][A_1(Fu) + B_1(Fu)]$$
$$+ \delta_2\delta_3G_3(t,t)\left[\int_0^1 G_3(v,v)G_2(v,v)dv\right][A_2(e_1(F))$$
$$+ B_2(e_1(F))] + \delta_3G_3(t,t)[A_3(e_2(F)) + B_3(e_2(F))]$$
$$\geq \delta_1\delta_2\delta_3G_3(t,t)(g_{32}g_{31} + g_{32}[A_2(G_1) + B_2(G_1)]$$
$$+ [A_3(G_2) + B_3(G_2)][g_{21} + A_2(G_1) + B_2(G_1)]$$
$$\cdot\left[\int_0^1 G_1(s,s)(Fu)(s)ds + A_1(Fu) + B_1(Fu)\right]$$
$$\geq \delta_1\delta_2\delta_3E(t)\frac{1}{C_1C_2C_3N_3}\|TFu\|_0$$
$$\geq E(t)\frac{\delta_1\delta_2\delta_3}{C_1C_2C_3N_3}(1-L)\|HFu\|_0$$
$$= \Theta E(t)\|Qu\|_0,$$

where $g_{ij} = \int_0^1 G_i(v,v)G_j(v,v)dv$ , $(i,j = 1,2,3,$ $i \neq j)$. So we have

$$(Qu)(t) \geq \Theta E(t)\|Qu\|_0. \quad (56)$$

Similarly, it is easy to see that

$$-(Qu)^{(2)}(t) \geq \Theta_2 E(t)\|(Qu)^{(2)}\|_0. \quad (57)$$

Indeed, using (34) $H$ can be expressed by

$$\begin{aligned} Hh &= (I + TG + (TG)^2 \\ &\quad + \ldots + (TG)^n + \ldots)Th \\ &= Th + TGTh + \ldots + (TG)^2Th \\ &\quad + \ldots + (TG)^nTh + \ldots \\ &= T(Ih + GTh + (GT)^2h \\ &\quad + \ldots + (GT)^nh + \ldots). \end{aligned} \quad (58)$$

If we differentiate the right side of (34) with help of (58), we have the following: $\forall\, h \in Y_+$,

$$T'(Ih + GTh + (GT)^2h + \ldots + (GT)^nh + \ldots)$$
$$= T'h + T'G(Th + (TG)Th + \ldots + (TG)^nTh + \ldots)$$
$$\leq T'h + T'G(Th + \|TG\|Th + \ldots + \|TG\|^nTh + \ldots)$$
$$\leq T'h + T'G(1 + L + \ldots + L^n + \ldots)Th$$
$$= T'h + \frac{1}{1-L}(T'G)Th.$$

Then the series

$$T'h + T'GTh + T'G(TG)Th + \ldots + T'G(TG)^nTh + \ldots$$

converges uniformly on $(0,1)$.

Using Lemma 10, if we differentiate both sides of (34), we get

$$(Hh)' = T'h + T'GTh + T'G(TG)Th + \ldots + T'G(TG)^nTh + \ldots \quad (59)$$

Similarly, using Lemma 10 it is also seen that

$$(Hh)^{(2)} = T^{(2)}h + T^{(2)}GTh + T^{(2)}G(TG)Th + \ldots + T^{(2)}G(TG)^nTh + \ldots, \quad (60)$$

because the series

$$T^{(2)}h + T^{(2)}GTh + T^{(2)}G(TG)Th + \ldots + T^{(2)}G(TG)^nTh + \ldots$$

also converges uniformly on $(0,1)$. If we differentiate both sides of (59), we find (60).

Finally, we differentiate twice both sides of equation (26) with respect to $t$ in order to find $T^{(2)}$

$$\begin{aligned} (Th)^{(2)}(t) &= \lambda_2(Th)(t) \\ &\quad - \int_0^1\int_0^1 G_3(t,\tau)G_1(\tau,s)h(s)dsd\tau \\ &\quad - \int_0^1 G_3(t,\tau)[A_1(h)\varphi_1(\tau) \\ &\quad + B_1(h)\psi_1(\tau)]d\tau \\ &\quad - [A_3(T_1(h))\varphi_3(v) + B_3(T_1(h))\psi_3(v)] \\ &= \lambda_2(Th)(t) - (T_3T_1h)(t).. \end{aligned} \quad (61)$$

Using (60) and (61) we obtain

$$(Hh)^{(2)} = \lambda_2(Hh)(t) - (H_1h)(t)$$

where $(Hh)(t)$ and $(H_1h)(t)$ is in (34) and (50), respectively. Let $h(t) = F(u)$, then we obtain

$$\begin{aligned}(Qu)^{(2)}(t) &= (HF(u))^{(2)} \\ &= \lambda_2(HF(u))(t) - (H_1F(u))(t).\end{aligned}$$

The proof of (57) is similar to the proof of (56), so we omit it.

So, $QP \subset P$.

Step 5. Let $d_2 = \min_{\frac{1}{4}\le t\le\frac{3}{4}} E(t)$, then $d_2 > 0$, and let $\Lambda = \Theta d_2$. Thus $\forall\, u \in P,\ \ u(t) \ge \Theta d_2 \|u\|_0 = \Lambda \|u\|_0\,,\ \ t \in \left[\frac{1}{4}, \frac{3}{4}\right].$

By

$$\lim_{|u|+|v|\to 0+} \inf \min_{t\in[0,1]} \left(\frac{f(t,u,v)}{|u|+|v|}\right) > \Gamma,$$

we can choose $\varepsilon > 0$ such that $\lim_{|u|+|v|\to 0+} \inf \min_{t\in[0,1]}(\frac{f(t,u,v)}{|u|+|v|}) > \Gamma + \varepsilon$.

Then $\exists\, r > 0$ such that $f(t,x,y) > (\Gamma+\varepsilon)(|x|+|y|)\ \ t \in [0,1]\,,\ 0 < |x|+|y| < (\sigma+1)r$. Let $\Omega_r = \left\{u \in P : \left\|u^{(2)}\right\|_0 < r\right\}$. For any $u \in \partial\Omega_r$, we have $\left\|u^{(2)}\right\|_0 = r,\ 0 < u(t) \le \|u\|_0 \le \sigma r, t \in (0,1)$, and so $f(t,u(t),u^{(2)}(t)) > (\Gamma+\varepsilon)(u(t)+|u^{(2)}(t)|)$, $t \in (0,1)$. Let $d_3 = \min_{\frac{1}{4}\le t\le\frac{3}{4}} E_2(t)$, then $d_3 > 0$, and let $\delta = \Theta_2 d_3$.

By $|u^{(2)}(t)| \ge \delta \left\|u^{(2)}\right\|_0 = \delta r, t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, it follows that

$$\begin{aligned}f(t,u(t),u^{(2)}(t)) &> (\Gamma+\varepsilon)(u(t)+|u^{(2)}(t)|) \\ &\ge (\Gamma+\varepsilon)|u^{(2)}(t)| \\ &\ge\ge (\Gamma+\varepsilon)\delta\|u^{(2)}\|_0,.\end{aligned}$$

where $t \in [\frac{1}{4}, \frac{3}{4}]$.

Step 6. Now we shall prove $\inf_{u\in\partial\Omega_r} \|(Qu)^{(2)}\|_0 > 0$. For any $u \in \partial\Omega_r$, by (37) we have

$$\begin{aligned}\left\|(Qu)^{(2)}\right\|_0 &\ge \frac{1}{\sigma}\|Qu\|_0 \\ &\ge \frac{1}{\sigma} Qu\left(\frac{1}{2}\right) \\ &\ge \frac{1}{\sigma}(TFu)\left(\frac{1}{2}\right) \\ &= \frac{1}{\sigma}\int_0^1\int_0^1\int_0^1 G_3\left(\frac{1}{2}, v\right) G_2(v,\tau) \\ &\quad G_3(\tau,s) f(s,u(s),u^{(2)}(s))ds d\tau dv \\ &\quad + \frac{1}{\sigma}\int_0^1\int_0^1 G_3\left(\frac{1}{2}, v\right) G_2(v,\tau)[A_1(f)\varphi_1(\tau) \\ &\quad + B_1(f)\psi_1(\tau)]d\tau dv \\ &\quad + \frac{1}{\sigma}\int_0^1 G_3(\frac{1}{2}, v)[A_2(T_1(f))\varphi_2(v) \\ &\quad + B_2(T_1(f))\psi_2(v)]dv \\ &\quad + \frac{1}{\sigma} A_3(T_2T_1(f))\varphi_3\left(\frac{1}{2}\right) \\ &\quad + \frac{1}{\sigma} B_3(T_2T_1(f))\psi_3\left(\frac{1}{2}\right) \\ &\ge \frac{1}{\sigma}\int_0^1\int_0^1\int_0^1 G_3\left(\frac{1}{2}, v\right) G_2(v,\tau) \\ &\quad G_3(\tau,s) f(s,u(s),u^{(2)}(s))ds d\tau dv \\ &\ge \frac{1}{\sigma}\delta_1\delta_2\delta_3 G_3\left(\frac{1}{2},\frac{1}{2}\right) m_{32}m_{21} \\ &\quad \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) f(s,u(s),u^{(2)}(s))ds \\ &\ge \frac{1}{\sigma}\delta_1\delta_2\delta_3 G_3\left(\frac{1}{2},\frac{1}{2}\right) m_{32}m_{21} \\ &\quad C_0(\Gamma+\varepsilon)\delta r > 0.\end{aligned} \tag{62}$$

Therefore, $\inf_{u\in\partial\Omega_r} \left\|(Qu)^{(2)}\right\|_0 > 0$.

Next we shall prove $\forall u \in \partial\Omega_r, 0 < \kappa \le 1, Qu \ne \kappa u$.

Suppose the contrary, that $\exists u_0 \in \partial\Omega_r, 0 < \kappa_0 \le 1$, such that $Qu_0 = \kappa_0 u_0$. By (37) we get

$$\begin{aligned}u_0(t) \ge \kappa_0 u_0(t) &= (Qu_0)(t) \ge (TFu_0)(t) \\ &= T(f(t,u_0(t),u_0^{(2)}(t))),\ t \in [0,1]\,.\end{aligned}$$

Let $v_0 = T(f(t,u_0(t),u_0^{(2)}(t)))$. Then $u_0(t) \ge v_0(t)$ and $v_0(t)$ satisfies the following BVP:

$$\begin{aligned}&-v_0^{(6)} + av_0^{(4)} + bv_0^{(2)} + cv_0 \\ &\quad = f(t,u_0(t),u_0^{(2)}(t)),\ 0 < t < 1.\end{aligned} \tag{63}$$

Multiplying (63) by $\sin\pi t$ and integrating on $[0,1]$ together with

$$\begin{aligned}v_0^{(2i-2)}(0) &= \sum_{i=1}^{m-2} a_i v_0^{(2i-2)}(\xi_i),\ i = 1,2,3 \\ v_0^{(2i-2)}(1) &= \sum_{i=1}^{m-2} b_i v_0^{(2i-2)}(\xi_i),\ i = 1,2,3\end{aligned}$$

and $u_0(t) \ge v_0(t)$, we get

$$\begin{aligned}&\Gamma\int_0^1 \sin\pi t v_0(t)ds + \pi((b - a\pi^2 - \pi^4) \\ &\sum_{i=1}^{m-2}(a_i+b_i)v_0(\xi_i) + (a\pi+\pi^3) \\ &\sum_{i=1}^{m-2}(a_i+b_i)v_0^{(2)}(\xi_i) - \sum_{i=1}^{m-2}(a_i+b_i)v_0^{(4)}(\xi_i)) \\ &= \int_0^1 \sin\pi t f(t,u_0(t),u_0^{(2)}(t))dt.\end{aligned}$$

It is easy to see that $b-a\pi^2-\pi^4<0, a\pi+\pi^3>0$, and $v_0(\xi_i)\geq 0, v_0^{(2)}(\xi_i)\leq 0, v_0^{(4)}(\xi_i)\geq 0$, it follows:

$$\Gamma\int_0^1 \sin\pi t v_0(t)dt \geq \int_0^1 \sin\pi t f(t,u_0(t),u_0^{(2)}(t))dt. \tag{64}$$

By $f(t,u_0(t),u_0^{(2)}(t)) > (\Gamma+\varepsilon)(|u_0(t)|+|u_0^{(2)}(t)|),\ t\in(0,1)$, we have

$$\begin{aligned}\Gamma\int_0^1 \sin\pi t u_0(t)dt &\geq \Gamma\int_0^1 \sin\pi t v_0(t)dt\\ &\geq \int_0^1 \sin\pi t f(t,u_0(t),u_0^{(2)}(t))dt\\ &\geq (\Gamma+\varepsilon)\int_0^1 \sin\pi s(|u_0(t)|+|u_0^{(2)}(t)|)dt\\ &\geq (\Gamma+\varepsilon)\int_0^1 \sin\pi t u_0(t)dt.\end{aligned}$$

Since $\int_0^1 \sin\pi s u_0(s)ds > 0$, we have $\Gamma\geq(\Gamma+\varepsilon)$, a contradiction.

We obtain $i(Q,\Omega_r,P)=0$.

Step 7. By $\lim_{|v|\to+\infty}\sup\max_{t\in[0,1]}\sup_{u\in[0,\infty)}\left(\frac{f(t,u,v)}{|v|}\right)<\Gamma_1$, we choose $0<\varepsilon<\Gamma_1$ such that $\lim_{|v|\to+\infty}\sup\max_{t\in[0,1]}\sup_{u\in[0,\infty)}(\frac{f(t,u,v)}{|v|})<(\Gamma_1-\varepsilon)$. Then $\exists R_0$, for $|y|\geq R_0$, $f(t,x,y)<(\Gamma_1-\varepsilon)|y|, t\in[0,1]$.

Let $\widehat{M}=\sup_{(t,x,|y|)\in[0,1]\times[0,\infty]\times[0,R_0]} f(t,x,y)$.

Then

$$f(t,x,y)<(\Gamma_1-\varepsilon)|y|+\widehat{M},$$

$\forall\, t\in[0,1],\ x\in[0,\infty), |y|\in[0,\infty)$.

Take $R>\max\left\{r,\frac{\widehat{M}}{\varepsilon}\right\}$. Putting

$$\Omega_R=\left\{u\in P:\left\|u^{(2)}\right\|_0<R\right\},$$

we next prove $\forall u\in\partial\Omega_R, \nu\geq 1, \nu u\neq Qu$.

Assume on the contrary that $\exists\nu_0\geq 1,\ u_0\in\partial\Omega_R$, $\nu_0 u_0\neq Qu_0$.

By (38) we get

$$\begin{aligned}u_0(t)&\leq \nu_0u_0(t)=(Qu_0)(t)=(HFu_0)(t)\\ &\leq\frac{1}{1-L}(TFu_0)(t)\leq\frac{1}{1-L}C_1C_2C_3N_3\\ &\left[\int_0^1 G_1(s,s)(Fu_0)(s)ds+A_1(Fu_0)+B_1(Fu_0)\right]\\ &\leq\frac{1}{1-L}C_1C_2C_3N_3\left[(\Gamma_1-\varepsilon)\left\|u_0^{"}\right\|_0+\widehat{M}\right]\\ &\left[\int_0^1 G_1(s,s)ds+A_1(1)+B_1(1)\right]\\ &\leq\frac{C_1C_2C_3N_1N_3}{1-L}(\Gamma_1-\varepsilon)\|u_0^{(2)}\|_0\\ &+\frac{C_1C_2C_3N_1N_3\widehat{M}}{1-L}\\ &=\left(1-\frac{\varepsilon}{\Gamma_1}\right)\|u_0\|_0+\frac{\widehat{M}}{\Gamma_1},\quad t\in[0,1].\end{aligned}$$

Hence,

$$\|u_0\|_0\leq\frac{C_1C_2C_3N_1N_3}{1-L}(\Gamma_1-\varepsilon)\|u_0^{(2)}\|_0+\frac{C_1C_2C_3N_1N_3\widehat{M}}{1-L},$$

and using $\frac{1}{\sigma}\|u_0^{(2)}\|_0\leq\|u_0\|_0$ we have

$$\|u_0^{(2)}\|_0\leq\left(1-\frac{\varepsilon}{\Gamma_1}\right)\|u_0^{(2)}\|_0+\frac{\widehat{M}}{\Gamma_1}, \tag{65}$$

where $\Gamma_1=\frac{1-L}{\sigma C_1C_2C_3N_1N_3}$.

By (65), we have $R=\left\|u_0^{(2)}\right\|_0\leq\frac{\widehat{M}}{\varepsilon}$, which is contradicts to $R>\frac{\widehat{M}}{\varepsilon}$.Then $i(Q,\Omega_R,P)=1$. In terms of the fixed index theory, we have $i(Q,\Omega_r,P)=0$, and so $i(Q,\Omega_R\backslash\bar{\Omega}_r,P)=1$. Thus BVP (1-2) has a positive solution. This completes the proof. □

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