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ABOUT A GENERALIZED CLASS OF CLOSE-TO-CONVEX FUNCTIONS DEFINED BY THE q-DIFFERENCE OPERATOR

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Abstract

In this paper we generalize the class of close-to-convex functions by the q-difference operator, for functions with negative coefficients and we study some properties of this generalized class. An analogue of the Pólya-Schoenberg conjecture is proved.

Keywords: close-to-convex functions, q-derivative, Pólya-Schoenberg conjecture

1 Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unite disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{A}$ the class of the functions f, normalized by $f(0) = 0 = f'(0) - 1$, which are analytic in U.

We say that f is starlike in U if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain with respect to origin. It is well-known that $f \in \mathcal{A}$ is starlike in U if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \text{ for all } z \in U.$$

We denote by S^* the class of starlike functions.

Let $\mathcal{K}$ be the class of convex functions. We say that $f \in \mathcal{A}$ is convex in U, if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a convex domain in $\mathbb{C}$. It is known that the function $f \in \mathcal{A}$ is convex in U if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

We say that $f \in \mathcal{A}$ is close-to-convex in U, if there is a convex function $g \in \mathcal{K}$ for which

$$\text{Re}\,\frac{f'(z)}{g'(z)} > 0,\ z \in U.$$

Since if $g \in \mathcal{K}$ then $zg' \in S^*$, an equivalent definition of close-to-conexity is the following.

The function $f \in \mathcal{A}$ is close-to-convex in U if there exist a starlike function $g \in S^*$ for which

$$\text{Re}\,\frac{zf'(z)}{g(z)} > 0,\ z \in U.$$

We denote by C the class of close-to-convex functions.

Let T denote a subclass of $\mathcal{A}$ consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \qquad (1)$$

where $a_j \geq 0$, $j = 2, 3, ...$ and $z \in U$. A function $f \in T$ is called a function with negative coefficients. If $f \in T$ and f is univalent, the followings are equivalent [9]:

(i) $\sum\limits_{j=2}^{\infty} ja_j \leq 1$,

(ii) $f \in T$,

(iii) $f \in T^*$, where $T^* = T \cap S^*$.

In our paper we generalize the class of close-to-convex functions, for functions with negative coefficients, and we obtain some intresting results on this generalized class.

In 1973, Ruscheweyh and Sheil-Small [7] proved the Pólya-Schoenberg conjecture, namely if ϕ is convex and $f \in S^*$ or K, then $f * \phi$ also belong to S^* or K. In our paper we prove the analogue of this conjecture for the generalized class of close-to-convex functions.

2 Preliminaries

To prove our main results we need the following preliminary definitions and theorems.

Let

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j$$

and

$$g(z) = z - \sum_{j=2}^{\infty} b_j z^j,$$

then the Hadamard product or the convolutions of the functions f and g is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z).$$

In 1908 Jackson introduced the Euler-Jackson q-difference oprator.

For $f \in \mathcal{A}$ of the form (1) and $0 < q < 1$, the q-derivative of the function f is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \tag{2}$$

where $z \neq 0$ and $D_q f(0) = f'(0)$.

From (2) we can deduce that

$$D_q f(z) = 1 - \sum_{j=2}^{\infty} \frac{1-q^j}{1-q} a_j z^{j-1},$$

where $z \neq 0$.

We note with $[z_0, z_1, ..., z_n; f]$ the divided differences at a system of distinct points $z_0, z_1, ..., z_n$, where

$$[z_0, z_1, ..., z_n; f] = \sum_{j=0}^{n} \frac{f(z_j)}{(z_j - z_0) \cdot ... \cdot (z_j - z_n)}.$$

The n-th order q derivative of the function f is defined in [8] as follows:

$$(D_q^n f)(z) = [n]_q [z, qz, ..., q^n z; f], \tag{3}$$

where $[n]_q = \dfrac{q^n - 1}{q - 1}$.

The q-derivative have several intresting application in quantum mechanics and generally in physics.

Theorem 2.1. *[10] Let $f : U_q \subseteq \mathbb{C} \to \mathbb{C}$ be q-derivable of order n, then*

$$(D_q^n f)(z) = (q-1)^{-n} z^{-n} q^{-C_n^2} \cdot$$

$$\cdot \sum_{j=0}^{n} \frac{[n]_q!}{[j]_q! [n-j]_q!} (-1)^j q^{C_j^2} f(q^{n-j} z).$$

A generalization of the class of close-to-convex functions by D_q difference operator we can found in [8]. In followings are given some generalizations of the class of close-to-convex functions, in the case of functions with negative coefficients, using the q-difference operator.

Definition 2.1. *A function $f \in T$ is said to be in the generalized class of close-to-convex functions of order γ, denoted by $UCC_q(\gamma)$, if*

$$\operatorname{Re} \frac{z D_q f(z)}{g(z)} \geq \gamma,$$

where $0 \leq \gamma < 1$ and $g \in T^*$.

Remark 2.1. *If $\gamma = 0$ then $UCC_q(0) = UCC_q$.*

Definition 2.2. *A function $f \in T$ is said to be in the generalized class of close-to-convex functions of order γ, relative to a fixed function $g \in T^*$, denoted by $UCC_q(g, \gamma)$, if*

$$\operatorname{Re} \frac{z D_q f(z)}{g(z)} \geq \gamma,$$

where $0 \leq \gamma < 1$.

Definition 2.3. *A function $f \in T$ is said to be in the class UCC_q^n, if*

$$\operatorname{Re} \frac{z D_q^n f(z)}{g(z)} > 0,$$

where $g \in T^*$.

3 Main results

Theorem 3.1. *Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \in \{2, 3, ...\}$ and $0 \leq \gamma < 1$.*
If $f \in UCC_q(\gamma)$, then there exist $g \in T^*$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, such that

$$\sum_{j=2}^{\infty} \left(\frac{1-q^j}{1-q} a_j - \gamma b_j \right) < 1 - \gamma. \tag{4}$$

If

$$\sum_{j=2}^{\infty} \frac{1-q^j}{1-q} a_j < 1 - \gamma, \tag{5}$$

then $f \in UCC_q(\gamma)$.
In the particular case when

$$\frac{1-q^j}{1-q} a_j \leq b_j, \ j \in \{2, 3, ...\}$$

the inequality (4) is necessary and sufficient condition for f to belongs to $UCC_q(\gamma)$.

Proof. Let $f \in UCC_q(\gamma)$, then there exists $g \in T^*$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$ such that

$$\operatorname{Re} \frac{z D_q f(z)}{g(z)} > \gamma, \ z \in U.$$

If $z \in [0,1)$, we obtain

$$\frac{z - \sum_{j=2}^{\infty} \frac{1-q^j}{1-q} a_j z^j}{z - \sum_{j=2}^{\infty} b_j z^j} > \gamma. \tag{6}$$

We note that for $z \in [0,1)$ we have $z - \sum_{j=2}^{\infty} b_j z^j > 0$, because $g \in T^*$ and in this case $\sum_{j=2}^{\infty} j b_j \leq 1$, then $\sum_{j=2}^{\infty} b_j < 1$.

The relation (6) is equivalent to

$$\sum_{j=2}^{\infty} \Big(\frac{1-q^j}{1-q} a_j - \gamma b_j \Big) z^{j-1} < 1 - \gamma.$$

For (5) we chose $g(z) = z$. Then

$$\gamma - \operatorname{Re} \Big(\frac{z D_q f(z)}{g(z)} - 1 \Big) < 1$$

is true if

$$\gamma + |D_q f(z) - 1| < 1,$$

but we have

$$\begin{aligned} \gamma + |D_q f(z) - 1| &\leq \sum_{j=2}^{\infty} \Big| \frac{1-q^j}{1-q} a_j \Big| + \gamma \\ &= \sum_{j=2}^{\infty} \frac{1-q^j}{1-q} a_j + \gamma. \end{aligned}$$

To prove the particular case, we suppose

$$\frac{1-q^j}{1-q} a_j \geq b_j, \ j \in \{2, 3, ...\}.$$

Then we have

$$\begin{aligned} &\gamma + \frac{\sum_{j=2}^{\infty} \Big| b_j - \frac{1-q^j}{1-q} a_j \Big| |z|^{j-1}}{1 - \sum_{j=2}^{\infty} b_j |z|^{j-1}} \\ &\leq \gamma + \frac{\sum_{j=2}^{\infty} \Big(\frac{1-q^j}{1-q} a_j - b_j \Big)}{1 - \sum_{j=2}^{\infty} b_j} \\ &= \frac{\gamma + \sum_{j=2}^{\infty} \Big(\frac{1-q^j}{1-q} a_j - \gamma b_j \Big)}{1 - \sum_{j=2}^{\infty} b_j} < 1, \end{aligned}$$

if we suppose that the inequality (4) is true. □

Theorem 3.2. *Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $g \in T^*$, where $a_j, b_j \geq 0$, $j \in \{2, 3, ..\}$ and $\gamma \in [0, 1)$.*
If $f \in UCC_q(g, \gamma)$, then

$$\sum_{j=2}^{\infty} \Big(\frac{1-q^j}{1-q} a_j - \gamma b_j \Big) < 1 - \gamma. \tag{7}$$

If

$$\sum_{j=2}^{\infty} \Big[\frac{1-q^j}{1-q} a_j + (2-\gamma) b_j \Big] < 1 - \gamma, \tag{8}$$

then $f \in UCC_q(g, \gamma)$.
In the particular case when

$$\frac{1-q^j}{1-q} a_j \leq b_j, \ j \in \{2, 3, ..\},$$

then (7) implies that $f \in UCC_q(g, \gamma)$.

Remark 3.1. *When $f_2(z) = z - \frac{z^2}{1+q} \in UCC_q(g_2, \gamma)$, where $g_2(z) = z - \frac{z^2}{2} \in T^*$ we have*

$$\operatorname{Re} \frac{z D_q f_2(z)}{g_2(z)} = \operatorname{Re} \frac{z(1-z)}{z\big(1 - \frac{z}{2}\big)} = 2 \operatorname{Re} \frac{1-z}{2-z} > 0.$$

But

$$\sum_{j=2}^{\infty} \frac{1-q^j}{1-q} a_j + (2-\gamma) b_j = 1 + \frac{2-\gamma}{2} = 2 - \frac{\gamma}{2} \not< 1.$$

This show that (8) is only a sufficies condition.

Because in [7] the authors proved that the convolution of a starlike and a convex function is starlike, we can give the following theorem.

Theorem 3.3. *Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$ and let $\phi(z) = z - \sum_{j=2}^{\infty} c_j z^j$ convex in U, where $a_j, b_j, c_j \geq 0$, $j \in \{2, 3, ..\}$.*
If $f \in UCC_q(g, \gamma)$, where $\frac{1-q^j}{1-q} a_j \leq b_j$ for $j \in \{2, 3, ..\}$, then $f * \phi \in UCC_q(g, \gamma)$.

Proof. Let

$$(f * \phi)(z) = z - \sum_{j=2}^{\infty} a_j c_j z^j.$$

We know from Definition 2.2 that if $(f * \phi)(z) \in UCC_q(g, \gamma)$, then

$$\operatorname{Re} \frac{z D_q (f * \phi)(z)}{(g * \phi)(z)} > \gamma,$$

where $(g * \phi) \in T^*$ and $0 \leq \gamma < 1$.
Suppose $f \in UCC_q(g, \gamma)$. Then by Theorem 3.2 we have

$$\sum_{j=2}^{\infty} \Big[\frac{1-q^j}{1-q} a_j - \gamma b_j\Big] < 1-\gamma. \tag{9}$$

To finish our proof, we must to show

$$\sum_{j=2}^{\infty} \Big[\frac{1-q^j}{1-q} a_j c_j - \gamma b_j c_j\Big] < 1-\gamma.$$

Since $\phi \in T$, the above inequality is equivalent to

$$\sum_{j=2}^{\infty} |c_j| \Big[\frac{1-q^j}{1-q} a_j - \gamma b_j\Big] < 1-\gamma. \tag{10}$$

Because ϕ is convex, by the coefficient delimitation theorem for convex functions we have $|c_j| \leq 1$, for $j = 2, 3....$
Then from (10) we obtain

$$\sum_{j=2}^{\infty} |c_j| \Big[\frac{1-q^j}{1-q} a_j - \gamma b_j\Big] < \sum_{j=2}^{\infty} \Big[\frac{1-q^j}{1-q} a_j - \gamma b_j\Big] < 1-\gamma,$$

and the proof is done.

□

Theorem 3.4. *Let* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in T^*$ *and*

$$F(z) = I_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt, c \in \mathbb{N}^*.$$

If $f \in UCC_q(g, \gamma)$, *where* $\frac{1-q^j}{1-q} a_j \leq b_j$ *for* $j \in \{2, 3, ..\}$, *then* $F \in UCC_q(g, \gamma)$.

Proof. Suppose $f \in UCC_q(g, \gamma)$. Then by Theorem 3.2 we have

$$\sum_{j=2}^{\infty} \Big[\frac{1-q^j}{1-q} a_j - \gamma b_j\Big] < 1-\gamma.$$

We know that if f has the form (1) then

$$F(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j = z - \sum_{j=2}^{\infty} c_j z^j.$$

If the

$$\sum_{j=2}^{\infty} \Big[\frac{1-q^j}{1-q} \cdot \frac{c+1}{c+j} a_j - \gamma b_j\Big] < 1-\gamma$$

inequality is true, then according to Theorem 3.2 we know that $F \in UCC_q(g, \gamma)$.
Next we prove

$$\frac{1-q^j}{1-q} \cdot \frac{c+1}{c+j} a_j - \gamma b_j < \frac{1-q^j}{1-q} a_j - \gamma b_j, \text{ for } j \geq 2. \tag{11}$$

The inequality (11) is equivalent to

$$\frac{1-q^j}{1-q} a_j \Big(1 - \frac{c+1}{c+j}\Big) > 0,$$

which is true for all $c \in \mathbb{N}^*$ and $j \geq 2$, and the proof is done. □

4 Conclusion

The class of close-to-convex functions has an important role in geometric function theory and it was introduced by W. Kaplan.

In the Definition 2.1 we have generalized the class of close-to-convex functions, using the q-difference operator. In the Definition 2.2 we have generalized the class of close-to-convex functions relative to a fixed function $g \in T^*$.

For this two generalized class of functions, in Theorem 3.1 and Theorem 3.2 we have proved several coefficient inequalities. Using these inequalities we have proved an analogue of the Pólya-Schoenberg conjecture, and finally we have showed the preserving propertie of the Bernardi integral operator defined on the $UCC_q(g, \gamma)$ class.

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