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EXISTENCE OF POSITIVE SOLUTIONS FOR AN ELASTIC CURVED BEAM EQUATION

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Abstract

This paper investigates the existence of positive solutions for a sixth-order differential equations. By using the Leggett-Williams fixed point theorem we give some new existence results.

Keywords: positive solutions, fixed point theorem

1 Introduction

Boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics and the existence and multiplicity of positive solutions for such problems has become an important area of investigation in recent years; we refer the reader to [1-15] and the references therein. For example, the deformations of an elastic beam in the equilibrium state can be described as a boundary value problem of some fourth-order differential equations.

Recently, boundary value problems for fourth-order ordinary differential equations have been extensively studied. It is well known that the deformation of the equilibrium state, an elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem:

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \qquad (1)$$

Existence of solutions for problem (1) was established for example by Gupta [1,2], Liu [3], Ma [4], Ma et. al. [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

In 2003, Li [10] studied the existence of positive solutions for the two-point boundary value problem with two constant parameters

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t, u(t)), \quad 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = 0 \qquad (2)$$

under the conditions

(i) $f(t,u) : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous;

(ii) $\alpha, \beta \in R$ and $\beta < 2\pi^2$, $\alpha \geq -\frac{\beta^2}{4}$, $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$.

Recently, Yang [11] investigates the following boundary value problem

$$u^{(4)}(t) = g(t)\, f(t, u(t), u'(t)), \quad 0 < t < 1,$$

$$u(0) = u'(1) = u''(0) = u^{(3)}(1) = 0.$$

It is well known that the deformation of the equilibrium state, an elastic circular ring segment with its two ends simply supported can be described by a boundary value problem for a sixth-order ordinary differential equation [12]:

$$\begin{aligned} u^{(6)} + 2u^{(4)} + u'' = f(t,u), \quad 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) \\ = u^{(4)}(0) = u^{(4)}(1) = 0, \end{aligned}$$

However, there are only a handful of articles on this topic. There is a classical definition for the Green function of ordinary linear differential equations with homogenous boundary conditions. The concept has been generalized for a class of degenerate systems of linear differential equations by Szeidl in [12].

In this paper we shall discuss the existence of positive solutions for the elastic curved beam equation

$$-u^{(6)}+au^{(4)}+bu''+cu=f(t,u,u',u''),\quad 0<t<1$$
$$u(0)=u''(0)=u^{(4)}(0)=0,$$
$$u'(1)=u^{(3)}(1)=u^{(5)}(1)=0, \tag{3}$$

where $u(t)$ is the tangential displacement of the center line of the circulat arch and $a,b,c\in R$. The first boundary condition $u(0)=u''(0)=u^{(4)}(0)=0$ means that the left end of the curved beam is supported by sliding clamps. The boundary condition $u'(1)=u^{(3)}(1)=u^{(5)}(1)=0$ means that the right end of the curved beam is simply supported. Our results will generalize those established in [10,11]. For this, we shall assume the following conditions throughout:

(H1) $f(t,u):[0,1]\times[0,\infty)\times[0,\infty)\times(-\infty,0]\to[0,\infty)$ is continuous.

(H2) $\left(\frac{\pi}{2}\right)^6+a\left(\frac{\pi}{2}\right)^4-b\left(\frac{\pi}{2}\right)^2+c>0$, where $a,b,c\in R$, $a=\lambda_1+\lambda_2+\lambda_3>-\left(\frac{\pi}{2}\right)^2, b=-\lambda_1\lambda_2-\lambda_2\lambda_3-\lambda_1\lambda_3>0, c=\lambda_1\lambda_2\lambda_3<0$ and $\lambda_1\geq 0\geq\lambda_2>-\left(\frac{\pi}{2}\right)^2, 0\leq\lambda_3<-\lambda_2$.

Assumption (H2) involves a three-parameter nonresonance condition.

2 Preliminaries

Let $Y=C[0,1]$ and $Y_+=\{u\in Y: u(t)\geq 0, t\in[0,1]\}$. It is well known that Y is a Banach space equipped with the norm $\|u\|_0=\sup_{t\in[0,1]}|u(t)|$. Let $E=C^2[0,1]$. We need the space E equipped with the norm

$$\|u\|_2=\max\{\|u\|_0,\|u'\|_0,\|u''\|_0\}.$$

It is easy to show that E is complete with the norm $\|u\|_2$. Let K be a cone in the real Banach space E.

Let γ,φ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. For positive real numbers a,b,c and d, we define the following convex sets

$$K(\gamma,d)=\{u\in K:\gamma(u)<d\};$$
$$K(\gamma,\alpha,b,d)=\{u\in K: b\leq\alpha(u),\gamma(u)\leq d\};$$
$$K(\gamma,\varphi,\alpha,b,c,d)=\{u\in K: b\leq\alpha(u),\varphi(u)\leq c,\gamma(u)\leq d\};$$

and a closed set

$$L(\gamma,\psi,a,d)=\{u\in K: a\leq\psi(u),\gamma(u)\leq d\}.$$

Theorem 1. *(Leggett-Williams Fixed Point Theorem). Let K be a cone in a real Banach space E. Let γ,φ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u)\leq\lambda\psi(u)$ for $0\leq\lambda\leq 1$, such that for some positive numbers M and d,*

$$\alpha(u)\leq\psi(u)\qquad\text{and}\qquad\|u\|\leq M\gamma(u)$$

for all $u\in\overline{K(\gamma,d)}$. Suppose $A:\overline{K(\gamma,d)}\to\overline{K(\gamma,d)}$ is a completely continuous operator and exists a,b,c with $a<b$ such that

(A_1) $\{u\in K(\gamma,\varphi,\alpha,b,c,d):\alpha(u)>b\}\neq\emptyset$ and $\alpha(Au)>b$ for $u\in K(\gamma,\varphi,\alpha,b,c,d)$;

(A_2) $\alpha(Au)>b$ for $u\in K(\gamma,\alpha,b,d)$ with $\varphi(Au)>c$;

(A_3) $0\notin L(\gamma,\psi,a,d)$ and $\psi(Au)<a$ for $u\in L(\gamma,\psi,a,d)$ with $\psi(u)=a$.

Then A has at least three fixed points u_1,u_2,u_3 in $\overline{K(\gamma,d)}$ satisfying $\gamma(u_i)\leq d$ for $i=1,2,3$, $\alpha(u_1)>b$, $a<\psi(u_2)$, $\alpha(u_2)<b$ and $\psi(u_3)<a$.

For $h\in Y$, consider the following linear boundary value problem:

$$-u^{(6)}+au^{(4)}+bu''+cu=h(t),\quad 0<t<1$$
$$u(0)=u'(1)=u''(0)=u^{(3)}(1)=u^{(4)}(0)=u^{(5)}(1)=0, \tag{4}$$

where a,b,c satisfy the assumption

$$\left(\frac{\pi}{2}\right)^6+a\left(\frac{\pi}{2}\right)^4-b\left(\frac{\pi}{2}\right)^2+c>0 \tag{5}$$

and let $\Gamma=\left(\frac{\pi}{2}\right)^6+a\left(\frac{\pi}{2}\right)^4-b\left(\frac{\pi}{2}\right)^2+c$. The inequality (5) follows immediately from the fact that Γ is the first eigenvalue of the problem $-u^{(6)}+au^{(4)}+bu''+cu=\lambda u$, $u(0)=u'(1)=u''(0)=u^{(3)}(1)=u^{(4)}(0)=u^{(5)}(1)=0$ and $\phi_1(t)=\sin\frac{\pi}{2}t$ is the first eigenfunction, i.e. $\Gamma>0$. Because the set $l_1=\left\{(a,b,c):\left(\frac{\pi}{2}\right)^6+a\left(\frac{\pi}{2}\right)^4-b\left(\frac{\pi}{2}\right)^2+c=0\right\}$ is the first eigenvalue set of the three-parameter boundary value problem $-u^{(6)}+au^{(4)}+bu''+cu=0$, $u(0)=u'(1)=u''(0)=u^{(3)}(1)=u^{(4)}(0)=u^{(5)}(1)=0$, if (a,b,c) lies in l_1, then by the Fredholm alternative the existence of a solution of the boundary value problem (4) cannot be guaranteed.

Let $P(\lambda)=\lambda^2+\beta\lambda-\alpha$ where $\beta<2\left(\frac{\pi}{2}\right)^2, \alpha\geq 0$. It is easy to see that equation $P(\lambda)=0$ has two real roots $\lambda_1,\lambda_2=\frac{-\beta\pm\sqrt{\beta^2+4\alpha}}{2}$, with $\lambda_1\geq 0\geq\lambda_2>-\left(\frac{\pi}{2}\right)^2$. Let λ_3 be a number such that $0\leq\lambda_3<-\lambda_2$. In this case, (4) satisfies the following decomposition form:

$$\begin{aligned}&-u^{(6)}+au^{(4)}+bu''+cu\\&=(-\frac{d^2}{dt^2}+\lambda_1)(-\frac{d^2}{dt^2}+\lambda_2)(-\frac{d^2}{dt^2}+\lambda_3)u,\\&\qquad 0<t<1.\end{aligned} \tag{6}$$

It is obvious that $a=\lambda_1+\lambda_2+\lambda_3>-\left(\frac{\pi}{2}\right)^2, b=-\lambda_1\lambda_2-\lambda_2\lambda_3-\lambda_1\lambda_3>0, c=\lambda_1\lambda_2\lambda_3<0$. Indeed, if we substitute a, b, c for (5), we obtain

$$\begin{aligned}&\left(\frac{\pi}{2}\right)^6+(\lambda_1+\lambda_2+\lambda_3)\left(\frac{\pi}{2}\right)^4\\&\quad+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)\left(\frac{\pi}{2}\right)^2\\&\quad+\lambda_1\lambda_2\lambda_3>0,\end{aligned}\tag{7}$$

hence

$$\begin{aligned}&-\lambda_2(\left(\frac{\pi}{2}\right)^4+(\lambda_1+\lambda_3)\left(\frac{\pi}{2}\right)^2+\lambda_1\lambda_3)\\&<\left(\frac{\pi}{2}\right)^6+(\lambda_1+\lambda_3)\left(\frac{\pi}{2}\right)^4+\lambda_1\lambda_3\left(\frac{\pi}{2}\right)^2\end{aligned}$$

from the assumptions $\lambda_1+\lambda_3>0$, $\lambda_1\lambda_3>0$, and we obtain $\left(\frac{\pi}{2}\right)^4+(\lambda_1+\lambda_3)\left(\frac{\pi}{2}\right)^2+\lambda_1\lambda_3>0$, so $\lambda_2>-\left(\frac{\pi}{2}\right)^2$. So, $0\geq\lambda_2>-\left(\frac{\pi}{2}\right)^2$ is applicable. Similarly, from (7), we have

$$\begin{aligned}&-\lambda_1(\left(\frac{\pi}{2}\right)^4+(\lambda_2+\lambda_3)\left(\frac{\pi}{2}\right)^2+\lambda_2\lambda_3)\\&<\left(\frac{\pi}{2}\right)^6+(\lambda_2+\lambda_3)\left(\frac{\pi}{2}\right)^4+\lambda_2\lambda_3\left(\frac{\pi}{2}\right)^2\end{aligned}$$

from the assumptions $0\leq\lambda_3<-\lambda_2$, $-\left(\frac{\pi}{2}\right)^2<\lambda_2<0$, and we obtain $\left(\frac{\pi}{2}\right)^4+(\lambda_2+\lambda_3)\left(\frac{\pi}{2}\right)^2+\lambda_2\lambda_3>0$, so $\lambda_1>-\left(\frac{\pi}{2}\right)^2$, i.e. the assumption $\lambda_1\geq0$ is applicable. From (7), we have

$$\begin{aligned}&-\lambda_3(\left(\frac{\pi}{2}\right)^4+(\lambda_1+\lambda_2)\left(\frac{\pi}{2}\right)^2+\lambda_1\lambda_2)\\&<\left(\frac{\pi}{2}\right)^6+(\lambda_1+\lambda_2)\left(\frac{\pi}{2}\right)^4+\lambda_1\lambda_2\left(\frac{\pi}{2}\right)^2,\end{aligned}$$

in which $\left(\frac{\pi}{2}\right)^4+(\lambda_1+\lambda_2)\left(\frac{\pi}{2}\right)^2+\lambda_1\lambda_2=\left(\frac{\pi}{2}\right)^4+\lambda_1(\left(\frac{\pi}{2}\right)^2+\lambda_2)+\lambda_2\left(\frac{\pi}{2}\right)^2>0$ because $\left(\frac{\pi}{2}\right)^4+\lambda_2\left(\frac{\pi}{2}\right)^2>0$ and $\lambda_1(\left(\frac{\pi}{2}\right)^2+\lambda_2)>0$. So $\lambda_3>-\left(\frac{\pi}{2}\right)^2$, i.e. the assumption $0\leq\lambda_3<-\lambda_2$ is applicable.

Suppose that $G_i(t,s)(i=1,2,3)$ is the Green function associated with

$$-u''+\lambda_i u=0,\quad u(0)=u'(1)=0.\tag{8}$$

We need the following lemmas.

Lemma 1. *Let* $\omega_i=\sqrt{|\lambda_i|}$, *then* $G_i(t,s)(i=1,2,3)$ *can be expressed as*

(i) *when* $\lambda_i>0$,

$$G_i(t,s)=\begin{cases}\dfrac{\sinh\omega_i t\cosh\omega_i(s-1)}{\omega_i\cosh\omega_i}, & 0\leq t\leq s\leq1\\ \dfrac{\cosh\omega_i(t-1)\sinh\omega_i s}{\omega_i\cosh\omega_i}, & 0\leq s\leq t\leq1\end{cases}$$

(ii) *when* $\lambda_i=0$,

$$G_i(t,s)=\begin{cases}t, & 0\leq t\leq s\leq1\\ s, & 0\leq s\leq t\leq1\end{cases}$$

(iii) *when* $-\left(\frac{\pi}{2}\right)^2<\lambda_i<0$,

$$G_i(t,s)=\begin{cases}\dfrac{\sin\omega_i t\cos\omega_i(1-s)}{\omega_i\cos\omega_i}, & 0\leq t\leq s\leq1\\ \dfrac{\sin\omega_i s\cos\omega_i(1-t)}{\omega_i\cos\omega_i}, & 0\leq s\leq t\leq1\end{cases}.$$

It is easy to obtain for $\frac{\partial G_i(t,s)}{\partial t}$ that

(i) when $\lambda_i>0$, $\frac{\partial G_i(t,s)}{\partial t}$

$$=\begin{cases}\dfrac{\cosh\omega_i t\cosh\omega_i(s-1)}{\cosh\omega_i}s, & 0\leq t\leq s\leq1\\ \dfrac{\sinh\omega_i(t-1)\sinh\omega_i s}{\cosh\omega_i}s, & 0\leq s\leq t\leq1\end{cases}$$

(ii) when $\lambda_i=0$, $\frac{\partial G_i(t,s)}{\partial t}$

$$=\begin{cases}1, & 0\leq t\leq s\leq1\\ 0, & 0\leq s\leq t\leq1\end{cases}$$

(iii) when $-\left(\frac{\pi}{2}\right)^2<\lambda_i<0$, $\frac{\partial G_i(t,s)}{\partial t}$

$$=\begin{cases}\dfrac{\cos\omega_i t\cos\omega_i(1-s)}{\cos\omega_i}, & 0\leq t\leq s\leq1\\ \dfrac{\sin\omega_i s\sin\omega_i(1-t)}{\cos\omega_i}, & 0\leq s\leq t\leq1\end{cases}.$$

It is easy to see that $G_i(t,s)(i=1,2,3)$ has the following properties:

(i) $G_i(t,s)>0, \forall t,s\in(0,1)$;

(ii) $\frac{\partial G_i(t,s)}{\partial t}>0, \forall t,s\in(0,1)$.

Let

$$\begin{aligned}M_1&=\int_0^1\int_0^1\int_0^1 G_1(1,v)G_2(v,s)G_3(s,\tau)g(\tau)\,d\tau ds dv,\\ M_2&=\int_0^1\int_0^1\int_0^1\frac{\partial G_1(0,v)}{\partial t}G_2(v,s)G_3(s,\tau)g(\tau)\,d\tau ds dv\\ M_3&=|\lambda_2|\int_0^1\int_0^1\int_0^1 G_1(1,v)G_2(v,s)G_3(s,\tau)g(\tau)\,d\tau ds dv\\&\quad+\int_0^1\int_0^1 G_1(1,s)G_3(s,\tau)g(\tau)\,d\tau ds\\ M&=\max\{M_1,M_2,M_3\}\\ M_4&=\int_0^1\int_0^1\int_0^1 G_1(\frac{1}{2},v)G_2(v,s)G_3(s,\tau)g(\tau)\,d\tau ds dv\\ M_5&=\max_{t\in[0,1]}\int_0^1\int_0^1\int_0^1\frac{\partial G_1(t,v)}{\partial t}G_2(v,s)G_3(s,\tau)g(\tau)\,d\tau ds dv.\end{aligned}$$

Now, since

$$\begin{aligned}&-u^{(6)}+au^{(4)}+bu''+cu\\&=\left(-\frac{d^2}{dt^2}+\lambda_1\right)\left(-\frac{d^2}{dt^2}+\lambda_2\right)\left(-\frac{d^2}{dt^2}+\lambda_3\right)u\\&=\left(-\frac{d^2}{dt^2}+\lambda_2\right)\left(-\frac{d^2}{dt^2}+\lambda_1\right)\left(-\frac{d^2}{dt^2}+\lambda_3\right)u\\&=h(t),\end{aligned}\tag{9}$$

the solution of boundary value problem (4) can be expressed by

$$u(t)=\int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)d\tau ds dv,\quad t\in[0,1].\tag{10}$$

Thus, for every given $h\in Y$, the boundary value problem (4) has a unique solution $u\in C^6[0,1]$ which is given by (10).

We now define a mapping $T : C[0,1] \to C[0,1]$ by

$$(Th)(t) = \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau), \quad h(\tau)d\tau ds dv \qquad t \in [0,1]. \tag{11}$$

Throughout this article we shall denote $Th = u$ the solution of the linear boundary value problem (4).

Let us define the cone $K \subset E$ by

$$K = \{u \in E : u(t) \geq 0, u''(t) \leq 0, u^{(4)}(t) \geq 0, \\ u(0) = u'(1) = u''(0) = u^{(3)}(1) = u^{(4)}(0) \\ = u^{(5)}(1) = 0\}.$$

Lemma 2. *$T : K \to K$ is linear and completely continuous.*

Proof. It is easy to check the operator T is completely continuous, so we omit it. Next we will show that $T(K) \subset K$. Assume that $h \in Y_+$ and $u = Th$ is the solution the boundary value problem (4). It is clear that the operator T maps Y_+ into Y_+. Now for all $\forall h \in Y_+, u = Th \in Y_+, u(0) = u'(1) = u''(0) = u^{(3)}(1) = u^{(4)}(0) = u^{(5)}(1) = 0$. Using (9) it is easy to see that

$$-u'' + \lambda_i u = \int_0^1 \int_0^1 G_j(t,v)G_k(v,\tau)h(\tau)d\tau dv, \tag{12}$$

$t \in [0,1]$ and

$$u^{(4)} - (\lambda_i + \lambda_j)u'' + \lambda_i\lambda_j u = \int_0^1 G_k(t,v)h(v)dv, \tag{13}$$

$t \in [0,1]$, where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$.

The equality (12) with the assumption $\lambda_2 \leq 0$ implies that $u'' \leq 0$. Similarly, the equality (13) with the assumptions $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 \leq 0$ implies that $u^{(4)} \geq 0$. □

Now, we define the nonnegative continuous concave functional α, the nonnegative continuous convex functional γ, φ and the nonnegative continuous functional ψ on the cone K by

$$\gamma(u) = \max_{t\in[0,1]} |u''(t)|,$$

$$\varphi(u) = \max_{t\in[0,1]} |u(t)|,$$

$$\psi(u) = \max_{t\in[0,1]} |u'(t)|,$$

$$\alpha(u) = \min_{t\in[\frac{1}{2},1]} |u(t)|.$$

Lemma 3. *If $u \in K$, then $u(1) \leq u'(0) \leq |u''(1)|$.*

Proof. The proof of $u(1) \leq u'(0)$ is similar to the proof of Lemma 1.2. in [11], so we omit it. From the mean value theorem, there exists $\xi \in (0,1)$ such that $u''(\xi) = u'(1) - u'(0) = -u'(0)$. By the fact that $u^{(4)}(t) \geq 0$ on $[0,1]$ and $u^{(3)}(1) = 0$ we know $u^{(3)}(t) \leq 0$, on $[0,1]$, so $u''(\xi) \leq u''(1)$, i.e. $u'(0) \leq |u''(1)|$. The proof is complete. □

Corollary 1. *If $u \in K$, then $\frac{1}{2}\varphi(u) \leq \alpha(u) \leq \varphi(u) \leq \psi(u) \leq \gamma(u)$ and $\|u\| = \gamma(u)$.*

Proof. From the boundary conditions, we may conclude that $\gamma(u) = |u''(1)|, \psi(u) = u'(0)$ and $\varphi(u) = u(1)$. Thus $\varphi(u) \leq \psi(u) \leq \gamma(u)$. At the same time, combining the definitions of $\alpha(u), \varphi(u)$ and $u''(t) \leq 0$ on $[0,1]$ (u is concave on $[0,1]$), we have

$$\frac{1}{2}\varphi(u) \leq \alpha(u) \leq \varphi(u).$$

□

3 Main results

We always assume there exist constants a, b, d such that $0 < a < b \leq \frac{d}{2}\left(\frac{2}{\pi}\right)^2 < \frac{d}{2}$ and

(H_1) $\quad f(t,u,v,w) \leq \frac{d}{M}$ for

$$(t,u,v,w) \in [0,1] \times [0,d] \times [-d,d] \times [-d,0]$$

(H_2) $\quad f(t,u,v,w) > \frac{b}{M_4}$ for

$$(t,u,v,w) \in \left[\frac{1}{2},1\right] \times [b,2b] \times [-d,d] \times [-d,0]$$

(H_3) $\quad f(t,u,v,w) < \frac{d}{M_5}$ for

$$(t,u,v,w) \in [0,1] \times [0,a] \times [-a,a] \times [-d,0]$$

Theorem 2. *Assume that $(H_1) - (H_3)$ hold and suppose that g satisfies*

$$0 < \int_0^1 G_i(t,\tau)g(\tau)\,d\tau < \infty, \quad i = 1, 2, 3.$$

Then the boundary value problem (3) has at least three positive solutions u_1, u_2 and u_3 satisfying
$\max_{t\in[0,1]} |u_i''(t)| \leq d$ *for* $i = 1, 2, 3$,
$\min_{t\in[\frac{1}{2},1]} |u_1(t)| > b$,
$\max_{t\in[0,1]} |u_2'(t)| > a$, $\min_{t\in[\frac{1}{2},1]} |u_2(t)| < b$,
and
$\max_{t\in[0,1]} |u_3'(t)| < a$.

Proof. If $u \in \overline{K(\gamma,d)}$, then $\alpha(u) \leq \psi(u)$ and $\|u\| \leq M\gamma(u)$ are satisfied because of Corollary 5. Next, we will check that conditions in Theorem 1. are satisfied, respectively. If $u \in \overline{K(\gamma,d)}$ then $\gamma(u) = \max_{t\in[0,1]} |u''(t)| \leq d$, i.e. $-d \leq u''(t) \leq 0$. From Lemma 4, and the definitions of φ, ψ and γ, we know $0 \le u(t) \le d$ and $-d \le u'(t) \le d$. Therefore, we can apply condition (H_1) holds. On the other hand,

$$\begin{aligned}\varphi(Tu) &= \max_{t\in[0,1]} |(Tu)(t)| \\ &= \max_{t\in[0,1]} \left| \int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau) \right. \\ &\quad f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau ds dv| \\ &= \int_0^1\int_0^1\int_0^1 G_1(1,v)G_2(v,s)G_3(s,\tau)f(\tau,u(\tau), \\ &\quad u'(\tau),u''(\tau))d\tau ds dv \\ &\le= \frac{d}{M}\int_0^1\int_0^1\int_0^1 G_1(1,v)G_2(v,s)G_3(s,\tau) \\ &\quad g(\tau)\, d\tau ds dv \le d,\end{aligned}$$

and Tu is concave on $[0,1]$. Indeed, it is easy to obtain

$$\begin{aligned}(Tu)''(t) = \lambda_2 (Tu)(t) - \int_0^1\int_0^1 G_1(t,s)G_3(s,\tau) \\ f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau ds \le 0\end{aligned}$$

because $\lambda_2 \le 0$ and $(Tu)(t) \ge 0$. So, it is easy to see that

$$\begin{aligned}\psi(Tu) &= \max_{t\in[0,1]} |(Tu)'(t)| \\ &= \max\{(Tu)'(0),(Tu)'(1)\} = (Tu)'(0) \\ &= \int_0^1\int_0^1\int_0^1 \frac{\partial G_1(0,v)}{\partial t}G_2(v,s)G_3(s,\tau) \\ &\quad f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau ds dv \\ &\le \frac{d}{M}\int_0^1\int_0^1\int_0^1 \frac{\partial G_1(0,v)}{\partial t}G_2(v,s)G_3(s,\tau) \\ &\quad g(\tau)\, d\tau ds dv \le d.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\gamma(u) &= \max_{t\in[0,1]} |(Tu)''(t)| = \max_{t\in[0,1]} |\lambda_2 (Tu)(t) \\ &\quad - \int_0^1\int_0^1 G_1(t,s)G_3(s,\tau)f(\tau,u(\tau),u'(\tau), \\ &\quad u''(\tau))d\tau ds| \\ &= |(Tu)''(1)| = |\lambda_2|(Tu)(1) \\ &\quad + \int_0^1\int_0^1 G_1(1,s)G_3(s,\tau)f(\tau,u(\tau),u'(\tau), \\ &\quad u''(\tau))d\tau ds \\ &\le \frac{d}{M}\Bigg(|\lambda_2| \int_0^1\int_0^1\int_0^1 G_1(1,v)G_2(v,s) \\ &\quad G_3(s,\tau)g(\tau)\, d\tau ds dv \\ &\quad + \int_0^1\int_0^1 G_1(1,s)G_3(s,\tau)g(\tau)d\tau ds \Bigg) \le d.\end{aligned}$$

That is $T : \overline{K(\gamma,d)} \to \overline{K(\gamma,d)}$.

Next, we examine the condition (A_1). First, we choose $u(t) = 2b\sin\left(\frac{\pi}{2}t\right) \in K$. Then $u'(t) = 2b\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}t\right)$ and $u''(t) = -2b\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}t\right)$. So, it is not difficult to check that

$$\varphi(u) = \max_{t\in[0,1]} |u(t)| \le 2b,$$

and

$$\alpha(u) = \min_{t\in\left[\frac{1}{2},1\right]} |u(t)| = 2b\sin\left(\frac{\pi}{4}\right) = \sqrt{2}b > b.$$

It is easy to see that $u(t) \in K(\gamma,\varphi,\alpha,b,2b,d) = \{u \in K : b \le \alpha(u), \varphi(u) \le 2b, \gamma(u) \le d\}$,because

$$\gamma(u) = \max_{t\in[0,1]} |u''(t)| = 2b\left(\frac{\pi}{2}\right)^2 \le d.$$

So, we obtain that $\{u \in K(\gamma,\varphi,\alpha,b,2b,d) : \alpha(u) > b\} \ne \emptyset$. Thus, if $u \in K(\gamma,\varphi,\alpha,b,2b,d)$, then $b \le u(t) \le 2b$, $|u'(t)| \le d$ and $|u''(t)| \le d$ for $\frac{1}{2} \le t \le 1$. If $u \in K(\gamma,\varphi,\alpha,b,2b,d)$, then $\alpha(Tu) = Tu(\frac{1}{2})$ by the fact that $(Tu)'(t) \ge 0$ for $t \in [0,1]$. Hence by (H_2), we have

$$\begin{aligned}\alpha(Tu) &= \int_0^1\int_0^1\int_0^1 G_1\left(\frac{1}{2},v\right)G_2(v,s)G_3(s,\tau) \\ &\quad f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau ds dv > \\ &> \frac{b}{M_4}\int_0^1\int_0^1\int_0^1 G_1\left(\frac{1}{2},v\right)G_2(v,s)G_3(s,\tau) \\ &\quad g(\tau)\, d\tau ds dv = b.\end{aligned}$$

So, assumption (A_1) of Theorem 1 holds.

Using Corollary 8. and $b \le \frac{d}{2}\left(\frac{2}{\pi}\right)^2 < \frac{d}{2}$, for $u \in K(\gamma,\alpha,b,d)$, $\varphi(Tu) > 2b$, we have

$$\alpha(Tu) \ge \frac{1}{2}\varphi(Tu) > b.$$

Thus, assumption (A_2) of Theorem 1 holds.

Finally, we check that (A_3) of Theorem 1 also holds. If $u(t) \equiv 0$, then $u'(t) \equiv 0$.Obviously, $\psi(u) = \max_{t\in[0,1]} |u'(t)| = 0 < a$, thus $0 \notin L(\gamma,\psi,a,d)$. If $u \in L(\gamma,\psi,a,d)$ with $\psi(u) = a$, then, $\gamma(u) = \max_{t\in[0,1]} |u''(t)| \le d$ we obtain $-d \le -u''(t) \le 0$. Moreover, by using Corollary 5 $\varphi(u) \le \psi(u)$, we have $0 \le u(t) \le a$.

By using (H_3), we have

$$\begin{aligned}\psi(Tu) &= \max_{t\in[0,1]} |(Tu)'(t)| \\ &= \max_{t\in[0,1]} \int_0^1\int_0^1\int_0^1 \frac{\partial G_1(t,v)}{\partial t}G_2(v,s)G_3(s,\tau) \\ &\quad f(\tau,u(\tau),u'(\tau),u''(\tau))d\tau ds dv \\ &< \frac{a}{M_5}\max_{t\in[0,1]} \int_0^1\int_0^1\int_0^1 \frac{\partial G_1(t,v)}{\partial t}G_2(v,s)G_3(s,\tau) \\ &\quad g(\tau)\, d\tau ds dv = a.\end{aligned}$$

Thus, (A_3) of Theorem 1 is satisfied. Therefore, Theorem 1 implies that boundary value problem (3) has at least three positive solutions u_1, u_2 and u_3. $\square$

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