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REMARKS ON A GENERALIZATION OF A QUESTION RAISED BY PÁL ERDŐS CONCERNING A GEOMETRIC INEQUALITY IN ACUTE TRIANGLES

Béla FINTA
“Petru Maior” University of Tîrgu Mureş
Nicolae Iorga Street, no.1, 540088, Tîrgu Mureş, Romania
e-mail: fintab@science.upm.ro

Abstract

The purpose of this paper is to give a negative answer to a possible generalization of an open question raised by Pál Erdős, concerning an inequality in acute triangles. We prove here that from $a < b < c$ does not follow $a^{2k} + l_a^{2k} < b^{2k} + l_b^{2k} < c^{2k} + l_c^{2k}$ in every acute triangle ABC, nor the opposite chain of inequalities, where $k \in \mathbb{N}, k \geq 2$, and a, b, c denotes the length of the triangles sites , while l_a, l_b, l_c denotes the length of the interior angle bisectors, as usual. We achieve this by constructing effectively two counterexamples, one for each type of inequalities.

Keywords: geometrical inequalities, acute triangle, interior bisectrices

1 Introduction

Let us consider the acute triangle ABC with sides $a = BC, b = AC$ and $c = AB$. In [1] appeared the following open question due to Pál Erdős: "if ABC is an acute triangle such that $a < b < c$ then $a + l_a < b + l_b < c + l_c$", where l_a, l_b, l_c means the length of the interior bisectrices corresponding to the sides BC, AC and AB, respectively.

In [2] Mihály Bencze proposed the following open question, which is a generalization of the problem of Pál Erdős: “determine all points $M \in Int(ABC)$, for which in case of $BC < CA < AB$ we have $CB + AA' < CA + BB' < AB + CC'$, where A’,B’,C’ is the intersection of AM, BM, CM with sides BC,CA,AB". Here with $Int(ABC)$ we denote the interior points of the triangle ABC.

If we try for “usual" acute triangles ABC, we can verify the validity of the Erdős inequality. But in [3] we realized to obtain an acute triangle for which the Erdős inequality is false: let ABC be such that $c = 10 + \epsilon, b = 10$ and $a = 1$, where $\epsilon > 0$ is a “very small" positive quantity. Using the trigonometrical way combined with some elementary properties from algebra and mathematical analysis we showed that for this “extreme" acute triangle from $a < b < c$ results $c + l_c < b + l_b$.

In [4] József Sándor proved some new geometrical inequalities using in the open question of Pál Erdős instead of bisectrices altitudes and medians.

In [5] Károly Dáné and Csaba Ignát studied the open question of Mihály Bencze using the computer.

In connection with Erdős’s problem we formulated the following open question: “if ABC is an acute triangle such that $a < b < c$ then $a^2 + l_a^2 < b^2 + l_b^2 < c^2 + l_c^2$". In [6] we proved the validity of this statement.

At the same time we formulated another new open question: “if ABC is an acute triangle such that $a < b < c$ then $a^4 + l_a^4 < b^4 + l_b^4 < c^4 + l_c^4$". In [7] we realized to find two acute triangles ABC such that in the first triangle from $a < b < c$ ($b = a + \epsilon, c = a + 2\epsilon$ with $\epsilon > 0$ a small positive quantity) we obtained $a^4 + l_a^4 < b^4 + l_b^4$, but in the second triangle from $a < b < c$ ($b = a + \epsilon, c = a\sqrt{2}$ with $\epsilon > 0$ a small positive quantity) we deduced $a^4 + l_a^4 > b^4 + l_b^4$. So the answer to our question is negative.

2 Main part

We can formulate the following more general open question using the problem of Pál Erdős: "if ABC is an acute triangle from $a < b < c$ results $a^\alpha + l_a^\alpha < b^\alpha + l_b^\alpha < c^\alpha + l_c^\alpha$, where $\alpha \in \mathbb{R}$. We mention, that this property is not obvious, because from $a < b < c$ results $l_a > l_b > l_c$. If $\alpha = 0$, then it is immediately that our problem is false.

The purpose of this paper is to study this open question for $\alpha = 2k, k \in \mathbb{N} - \{0\}$ even natural number.

Proposition 1. *There exist acute triangles ABC with $a < b < c$, such that for these triangles we have $a^{2k} + (l_a)^{2k} < b^{2k} + (l_b)^{2k}$ for every even natural number $2k, k \in \mathbb{N} - \{0\}$.*

Proof. We have the following sequence of equivalent inequalities: $a^{2k} + (l_a)^{2k} < b^{2k} + (l_b)^{2k} \Leftrightarrow (l_a)^{2k} - (l_b)^{2k} < b^{2k} - a^{2k} \Leftrightarrow (l_a^2 - l_b^2) \cdot [(l_a)^{2k-2} + (l_a)^{2k-4} \cdot (l_b)^2 + \cdots + (l_a)^2 \cdot (l_b)^{2k-4} + (l_b)^{2k-2}] < (b^2 - a^2) \cdot (b^{2k-2} + b^{2k-4} \cdot a^2 + \cdots + b^2 \cdot a^{2k-4} + a^{2k-2})$

But $l_a^2 = \frac{bc\cdot[(b+c)^2 - a^2]}{(b+c)^2}$ and $l_b^2 = \frac{ac\cdot[(a+c)^2 - b^2]}{(a+c)^2}$ so $l_a^2 - l_b^2 = bc - \frac{a^2bc}{(b+c)^2} - ac + \frac{ab^2c}{(a+c)^2} = c \cdot (b - a) + abc \cdot [\frac{b}{(a+c)^2} - \frac{a}{(b+c)^2}] = c \cdot (b-a) + abc \cdot \frac{(b-a)\cdot[(b^2+ba+a^2)+2c(b+a)+c^2]}{(a+c)^2\cdot(b+c)^2} = (b-a) \cdot [c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}]$.

This means, that $a^{2k} + (l_a)^{2k} < b^{2k} + (l_b)^{2k} \Leftrightarrow [c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}] \cdot [(l_a)^{2k-2} + (l_a)^{2k-4} \cdot (l_b)^2 + \cdots + (l_a)^2 \cdot (l_b)^{2k-4} + (l_b)^{2k-2}] < (b+a) \cdot (b^{2k-2} + b^{2k-4} \cdot a^2 + \cdots + b^2 \cdot a^{2k-4} + a^{2k-2})$.

At the first we choose the acute triangle ABC, such that $b = a + \epsilon$ and $c = a + 2\epsilon$, where $\epsilon > 0$ is a small positive quantity. Then $\lim_{\epsilon\to 0}(l_a)^2 = \lim_{\epsilon\to 0} \frac{bc\cdot[(b+c)^2-a^2]}{(b+c)^2} = \lim_{\epsilon\to 0} \frac{(a+\epsilon)(a+2\epsilon)\cdot[(a+\epsilon+a+2\epsilon)^2-a^2]}{(a+\epsilon+a+2\epsilon)^2} = \frac{3a^2}{4}$, $\lim_{\epsilon\to 0} l_a = \frac{\sqrt{3}a}{2}$,

$\lim_{\epsilon\to 0}(l_b)^2 = \lim_{\epsilon\to 0} \frac{ac\cdot[(a+c)^2-b^2]}{(a+c)^2} = \lim_{\epsilon\to 0} \frac{a\cdot(a+2\epsilon)\cdot[(a+a+2\epsilon)^2-(a+\epsilon)^2]}{(a+a+2\epsilon)^2} = \frac{3a^2}{4}$, $\lim_{\epsilon\to 0} l_b = \frac{\sqrt{3}a}{2}$, $\lim_{\epsilon\to 0} b = \lim_{\epsilon\to 0}(a+\epsilon) = a$, and $\lim_{\epsilon\to 0}[c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}] = \lim_{\epsilon\to 0}[(a+2\epsilon) + a \cdot (a+\epsilon) \cdot (a+2\epsilon) \cdot \frac{(a+\epsilon)^2+(a+\epsilon)\cdot a+a^2+2\cdot(a+2\epsilon)\cdot(a+\epsilon+a)+(a+2\epsilon)^2}{(a+a+2\epsilon)^2\cdot(a+\epsilon+a+2\epsilon)^2}] = \frac{3a}{2}$.

This means, that $\lim_{\epsilon\to 0}[c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}] \cdot [(l_a)^{2k-2} + (l_a)^{2k-4} \cdot (l_b)^2 + \cdots + (l_a)^2 \cdot (l_b)^{2k-4} + (l_b)^{2k-2}] \le \lim_{\epsilon\to 0}(b+a) \cdot (b^{2k-2} + b^{2k-4} \cdot a^2 + \cdots + b^2 \cdot a^{2k-4} + a^{2k-2})$, i.e. $\frac{3a}{2} \cdot (\frac{\sqrt{3}a}{2})^{2k-2} \cdot k \le 2a \cdot a^{2k-2} \cdot k$, i.e. $(\frac{\sqrt{3}}{2})^{2k} \le 1$.

We can see immediately that the inequality $(\frac{\sqrt{3}}{2})^{2k} < 1$ is true for all $k \in \mathbb{N} - \{0\}$. Using the definition of the limit we can conclude, that there exists a small positive real number $\epsilon_0 > 0$ such that for the triangle $A_0B_0C_0$ with $B_0C_0 = a, A_0C_0 = b = a + \epsilon_0$ and $A_0B_0 = c = a + 2\epsilon_0$ we obtain $a^{2k} + (l_a)^{2k} < b^{2k} + (l_b)^{2k}$ for every even natural number $2k, k \in \mathbb{N} - \{0\}$. □

Proposition 2. *There exist acute triangles ABC with $a < b < c$, such that for these triangles we have $a^{2k} + (l_a)^{2k} > b^{2k} + (l_b)^{2k}$ for every natural number $k \in \mathbb{N}, k \ge 2$.*

Proof. Using the above presented sequence of ideas we get similarly, that $a^{2k} + (l_a)^{2k} > b^{2k} + (l_b)^{2k} \Leftrightarrow [c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}] \cdot [(l_a)^{2k-2} + (l_a)^{2k-4} \cdot (l_b)^2 + \cdots + (l_a)^2 \cdot (l_b)^{2k-4} + (l_b)^{2k-2}] > (b+a) \cdot (b^{2k-2} + b^{2k-4} \cdot a^2 + \cdots + b^2 \cdot a^{2k-4} + a^{2k-2})$.

At the second we choose the acute triangle ABC, such that $b = a + \epsilon$ and $c = a\sqrt{2}$, where $\epsilon > 0$ is a small positive quantity. Then $\lim_{\epsilon\to 0}(l_a)^2 = \lim_{\epsilon\to 0} \frac{bc\cdot[(b+c)^2-a^2]}{(b+c)^2} = \lim_{\epsilon\to 0} \frac{(a+\epsilon)\cdot a\sqrt{2}\cdot[(a+\epsilon+a\sqrt{2})^2-a^2]}{(a+\epsilon+a\sqrt{2})^2} = (4 - 2\sqrt{2}) \cdot a^2$, $\lim_{\epsilon\to 0} l_a = \sqrt{4 - 2\sqrt{2}} \cdot a$,

$\lim_{\epsilon\to 0}(l_b)^2 = \lim_{\epsilon\to 0} \frac{ac\cdot[(a+c)^2-b^2]}{(a+c)^2} = \lim_{\epsilon\to 0} \frac{a\cdot a\sqrt{2}\cdot[(a+a\sqrt{2})^2-(a+\epsilon)^2]}{(a+a\sqrt{2})^2} = (4 - 2\sqrt{2}) \cdot a^2$, $\lim_{\epsilon\to 0} l_b = \sqrt{4 - 2\sqrt{2}} \cdot a$, $\lim_{\epsilon\to 0} b = \lim_{\epsilon\to 0}(a + \epsilon) = a$, and $\lim_{\epsilon\to 0}[c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}] = \lim_{\epsilon\to 0}[(a\sqrt{2}) + a \cdot (a+\epsilon) \cdot (a\sqrt{2}) \cdot \frac{(a+\epsilon)^2+(a+\epsilon)\cdot a+a^2+2\cdot(a\sqrt{2})\cdot(a+\epsilon+a)+(a\sqrt{2})^2}{(a+a\sqrt{2})^2\cdot(a+\epsilon+a\sqrt{2})^2}] = (16 - 10\sqrt{2}) \cdot a$.

This means, that $\lim_{\epsilon\to 0}[c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}] \cdot [(l_a)^{2k-2} + (l_a)^{2k-4} \cdot (l_b)^2 + \cdots + (l_a)^2 \cdot (l_b)^{2k-4} + (l_b)^{2k-2}] \ge \lim_{\epsilon\to 0}(b+a) \cdot (b^{2k-2} + b^{2k-4} \cdot a^2 + \cdots + b^2 \cdot a^{2k-4} + a^{2k-2})$, i.e. $(16 - 10 \cdot \sqrt{2}) \cdot a \cdot (\sqrt{4 - 2\sqrt{2}} \cdot a)^{2k-2} \cdot k \ge 2 \cdot a \cdot a^{2k-2} \cdot k$, i.e. $(16 - 10 \cdot \sqrt{2}) \cdot (4 - 2\sqrt{2})^{k-1} \ge 2$. But for $k \ge 2$ we have $(16 - 10 \cdot \sqrt{2}) \cdot (4 - 2\sqrt{2})^{k-1} \ge (16 - 10 \cdot \sqrt{2}) \cdot (4 - 2\sqrt{2}) > 2$.

So the inequality $(16 - 10 \cdot \sqrt{2}) \cdot (4 - 2\sqrt{2})^{k-1} > 2$ is true for all $k \in \mathbb{N}, k \ge 2$. Using the definition of the limit we can conclude, that there exists a small positive real number $\epsilon_1 > 0$ such that for the triangle $A_1B_1C_1$ with $B_1C_1 = a, A_1C_1 = b = a + \epsilon_1$ and $A_1B_1 = c = a\sqrt{2}$ we obtain $a^{2k} + (l_a)^{2k} > b^{2k} + (l_b)^{2k}$ for every even natural number $2k, k \in \mathbb{N}, k \ge 2$.

□

3 Discussion and conclusion

Conclusion Proposition 1 and 2 show that using the usual notations for the length of sides and the length of the interior angle bisectors of an acute triangle, if we suppose $a < b < c$ then does not result $a^{2k} + l_a^{2k} < b^{2k} + l_b^{2k} < c^{2k} + l_c^{2k}$, nor the opposite chain of the inequalities for any even values greater than 2 of the exponents.

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