Scientific Bulletin of the „Petru Maior" University of Tîrgu Mureş
Vol. 15 (XXXII) no. 1, 2018
ISSN-L 1841-9267 (Print), ISSN 2285-438X (Online), ISSN 2286-3184 (CD-ROM)

AN EXEMPLIFIED INTRODUCTION OF THE LAPLACE TRANSFORM

Marcel Bogdan
"Petru Maior" University of Tg. Mureş, Romania
e-mail: marcel.bogdan@science.upm.ro

ABSTRACT

We solve a linear Cauchy problem with discontinuous perturbation two ways, by solving continuous Cauchy problems successively and by using Laplace transform. An example is given when the last one cannot be used any longer, still the Cauchy problems are solvable and the Cauchy problem with discontinuous perturbation as well.

Keywords: Cauchy problem, Laplace transform, Lebesgue integrability, Fourier transform, harmonic oscillator.

1 Typical Cauchy problem

Laplace transform became common knowledge[1] with practical application especially in engineering (electronics; the example of the harmonic oscillator $y'' + \omega^2 y = f(x)$ is widely spread). It has the following technical explanation: a "signal" y is sent into an "amplifier", and get an "output signal" Y. In the present note we motivate its use a bit more. The theoretical part, treated with examples, exercises, and theorems, can be found for example in [2, 4, 9, 10, 12, 13, 14]. In particular, for the automatic systems theory one can consult [3].

This powerful mathematical tool has been used over the years in different studies, like generalizations on arbitrary time scales [1, 8], computing fundamental solutions for certain PDEs [5, 6], integral stability of certain linear processes [7], the abstract wave equation [11], Hyers-Ulam stability of a linear differential equation of the nth order [15].

Our purpose is to give an alternative to introduce the Laplace transform in order to have a natural meaning.

Let us consider the following typical Cauchy problem

$$(C) \quad \begin{cases} y'(x) - y(x) = f(x) \\ y(0) = 1, \end{cases}$$

where f is a given (not necessarily continuous) function.[2] [3]

In order to make a comparison between two approaches, i.e. by solving successively Cauchy problems and by using Laplace transform, we firstly chose $f(x) = \begin{cases} 1, & x \in [0,1) = I \\ 0, & x \in [1,+\infty) = J. \end{cases}$ Since f is not continuous, the problem does not have a classic solution, i.e. of the class C^1. On the interval I the problem

$$(C_I) \quad \begin{cases} y' - y = f \\ y(0) = 1, \end{cases}$$

has the classic solution $y_I(x) = 2 \cdot e^x - 1$. We go on to solve the problem on J

$$(C_J) \quad \begin{cases} y' - y = f \\ y(1) = y_I(1-) = 2e - 1, \end{cases}$$

finding the classic solution $y_J(x) = (2 - e^{-1}) \cdot e^x$.

[1] see $https://en.wikipedia.org/wiki/Laplace\_transform$ approximatively 1.420.000 results generated by $https://www.google.ro/$

[2] The solution of (C) is given by

$$y = y_0 + y_0 * f,$$

where $y_0 = \exp(\cdot)$ is the solution of the homogeneous equation and " $*$ " is the convolution operation.

[3] Let $f \in \mathcal{D}'(0,+\infty)$ be a given function, $a_k \in C^\infty[0,+\infty)$, $k \in \{1,...,n\}$, and $L = \sum_{k=1}^{n} a_k \cdot D^k$ a linear operator. It is known that the solution of the equation

$$Ly = f$$

is $y = E * f$, with E the fundamental solution. More, if $Ly_0 = 0$, then $y_0 + y$ is also solution.

Then, the generalized solution of (C) is

$$y(x)=\begin{cases} y_I(x), & x\in I\\ y_J(x), & x\in J,\end{cases} \tag{1}$$

with $x = 1$ angular point $y'(1-) = 2e$, $y'(1+)=2e-1$.

The Laplace transform is defined as $y\mapsto \mathcal{L}(y)=Y$,

$$\mathcal{L}(y)(s)=\int_0^\infty e^{-sx}\cdot y(x)\,dx \overset{not.}{=} Y(s),$$

if the integral exists, for a function $y:\mathbb{R}\to\mathbb{R}$ that satisfies the following conditions:

1° $y(x)=0, \forall x<0$;

2° on any interval $[0,N]$, y is continuous or has at most finitely number of discontinuity points (it is piecewise continuous);

3° there exists $M>0$ and $\sigma\in\mathbb{R}$ such that

$$|y(x)|\le M\cdot e^{\sigma x}, \forall x>N$$

(i.e., y has at most exponential growth).

From the usual computation one has $\mathcal{L}(e^{ax})(s)=\frac{1}{s-a}$, for $a\in\mathbb{R}$,[4] therefore the original Laplace $e^{ax}=\mathcal{L}^{-1}\left(\dfrac{1}{s-a}\right)$; also, $\mathcal{L}(y')(s)=s\cdot Y(s)-y(0)$.

Applied $\mathcal{L}$ to the equation in (C), by its linearity, one obtains

$$s\cdot Y(s)-1-Y(s)=-\frac{1}{s}\cdot(e^{-s}-1);$$

thus

$$Y(s)=-\frac{1}{s\cdot(s-1)}\cdot(e^{-s}-1)+\frac{1}{s-1}.$$

So, the original Laplace is given by

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(-\frac{1}{s\cdot(s-1)}\cdot(e^{-s}-1)+\frac{1}{s-1}\right)\\ &= -\mathcal{L}^{-1}\left(\frac{1}{s\cdot(s-1)}\cdot e^{-s}\right)\\ &\quad +\mathcal{L}^{-1}\left(\frac{1}{s\cdot(s-1)}\right)+\mathcal{L}^{-1}\left(\frac{1}{s-1}\right). \end{aligned} \tag{2}$$

We apply the following result.

[4]for $a = 0$ see $https://www.khanacademy.org/math/differential-equations/laplace-transform/laplace-transform-tutorial/v/laplace-transform-1)$

Theorem 1. ([13], Theorem 18) *If $\tilde{Y}$ is a Laplace transform and $\tilde{y}$ is its original, then*

$$\begin{aligned} \mathcal{L}^{-1}(e^{-as}\cdot\tilde{Y}(s)) &= \begin{cases}\tilde{y}(x-a), & x>a>0\\ 0, & x<a\end{cases}\\ &= \tilde{y}(x-a)\cdot u(x-a), \end{aligned} \tag{3}$$

where $u(\cdot)$ is the unit step function.

In particular, for $\tilde{Y}(s)=\dfrac{1}{s\cdot(s-1)}=\dfrac{1}{s-1}-\dfrac{1}{s}$, by (3) one has

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s\cdot(s-1)}\cdot e^{-s}\right) &= \begin{cases}\tilde{y}(x-1), & x>1\\ 0, & x<1\end{cases}\\ &= \begin{cases}e^{x-1}-1, & x>1\\ 0, & x<1.\end{cases} \end{aligned}$$

Consequently, from (2) one obtains

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}(Y(s))\\ &= -\begin{cases}e^{x-1}-1, & x>1\\ 0, & x<1\end{cases} + e^x-1+e^x\\ &= \begin{cases}(2-e^{-1})\cdot e^x, & x>1\\ 2\cdot e^x-1, & x<1,\end{cases} \end{aligned}$$

that is the solution given in (1) almost everywhere, i.e. except $x=1$.

Similarly, for the harmonic oscillator we proceed with basic computations.

Example 1. *Let $\omega > 0$, $f(x) = \begin{cases}1, & x\in[0,2\pi/\omega)=I_1\\ 0, & x\in[2\pi/\omega,+\infty)=I_2,\end{cases}$ and the following Cauchy problem*

$$(C_\omega)\quad \begin{cases} y''(x)+\omega^2 y(x)=f(x)\\ y(0)=0\\ y'(0)=1.\end{cases}$$

The problem on I_1 is (C_{I_1}) $\begin{cases} y''+\omega^2 y=1\\ y(0)=0\\ y'(0)=1,\end{cases}$ and has solution $y_{I_1}(x)=\frac{1}{\omega^2}\cdot(1-\cos\omega x)+\frac{1}{\omega}\cdot\sin\omega x$. On I_2 it becomes

$$(C_{I_2})\quad \begin{cases} y''+\omega^2 y=0\\ y(2\pi/\omega)=y_{I_1}(2\pi/\omega-)=0\\ y'(2\pi/\omega)=y'_{I_1}(2\pi/\omega-)=1,\end{cases}$$

and has solution $y_{I_2}(x)=\frac{1}{\omega}\cdot\sin\omega x$.

Then, the generalized solution of (C_ω) is

$$y(x)=\begin{cases} y_{I_1}(x), & x\in I_1\\ y_{I_2}(x), & x\in I_2.\end{cases} \tag{4}$$

Applied $\mathcal{L}$ to the equation in (C_ω), one has $\mathcal{L}(y'')+\omega^2\mathcal{L}(y)=\mathcal{L}(f)$ thus obtains $s^2\cdot Y(s)-1+\omega^2Y(s)=\mathcal{L}(f)$, and so $Y(s)=\frac{1}{s^2+\omega^2}\cdot[\mathcal{L}(f)+1]$. Since $\mathcal{L}(f)=-\frac{1}{s}\cdot(e^{-\frac{2\pi}{\omega}s}-1)$ it follows that the original Laplace is given by

$$\begin{aligned}
y(x) &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\bigg(-\frac{1}{s\cdot(s^2+\omega^2)}\cdot(e^{-\frac{2\pi}{\omega}s}-1) \\
&\qquad +\frac{1}{s^2+\omega^2}\bigg) \\
&= -\mathcal{L}^{-1}\bigg(\frac{1}{s\cdot(s^2+\omega^2)}\cdot e^{-\frac{2\pi}{\omega}s}\bigg) \\
&\quad +\mathcal{L}^{-1}\bigg(\frac{1}{s\cdot(s^2+\omega^2)}\bigg)+\mathcal{L}^{-1}\bigg(\frac{1}{s^2+\omega^2}\bigg) \\
&= \begin{cases} 0, & x\in I_1 \\ -\frac{1}{\omega^2}\cdot(1-\cos\omega x), & x\in I_2 \end{cases} \\
&\quad +\frac{1}{\omega^2}\cdot(1-\cos\omega x)+\frac{1}{\omega}\cdot\sin\omega x. \qquad (5)
\end{aligned}$$

Here we used the method given in [13].[5] *The solution given in* (5) *is a.e. with the one from* (4).

2 Some illustrative examples

By its definition Laplace transform cannot be used for any linear Cauchy problem.

Observation 1. *The solution of the Cauchy problem $\begin{cases} y'-y=(2x-1)\cdot e^{x^2} \\ y(0)=1, \end{cases}$ is $y(x)=e^{x^2}$, $x\in[0,+\infty)$ and can be obtained by Lagrange's method. The Laplace transform cannot be used, obviously because of the growth condition 3°, the function $f(x)=(2x-1)\cdot e^{x^2}$ being not a Laplace original. Anyway, suppose that there exist $M>0$ and $\sigma\in\mathbb{R}$ such that $|2x-1|\cdot e^{x^2}\le M\cdot e^{\sigma x},\forall x>N$. Take $x_0:=|\sigma|+N+M+1>N$, that is large enough to get the contradiction.*

In general, let $I\subseteq\mathbb{R}$, $x_0\in I$, and $(I_k)_{k=0}^{\infty}$ a partition of I, $I_k=[x_k,x_{k+1})$. Let $L=\sum_{k=1}^{n}a_k\cdot D^k$ be a linear operator, $a_k\in\mathbb{R}$, $y_{0j}, j\in\{0,...,n-1\}$ be given. A solution of the Cauchy problem $(C)\ \begin{cases} L(y)=f \\ y^{(j)}(x_0)=y_{0j}, \end{cases}\ j\in\{0,...,n-1\}$, can be obtained by solving successively Cauchy problems $(C_{I_k})\ \begin{cases} L(y)=f|_{I_k} \\ y^{(j)}(x_k)=y_{I_k}^{(j)}(x_k-), \end{cases}\ j\in\{0,...,n-1\}$, or by using the Laplace transform, when it is possible. In the following we exemplify ones more.

Example 2. *Let $n\in\mathbb{N}$ be fixed and $I_k=[k,k+1[$, $0\le k\le n$. Let us consider in (C) the function*

$$f(x)=-x+k+1,\ x\in\cup_{k=0}^{n}I_k.$$

Solving Cauchy problem on $I_0=[0,1[$ we get the classic solution $y_0(x)=e^x+x$. We go on to solve the problem on I_k, successively for $k\in\{1,...,n\}$

$$(C_{I_k})\quad \begin{cases} y'-y=f(x) \\ y(k)=y_{I_k}(k-), \end{cases}$$

finding the classic solution $y_{I_k}(x)=\sum_{i=0}^{k}e^{-i}\cdot e^x+x-k$. Then, the generalized solution of (C) is

$$y(x)=y_{I_k}(x),\ x\in\cup_{k=0}^{n}I_k, \qquad (6)$$

with $x=k$ angular points $y'(k-)=\sum_{i=0}^{k-1}e^{-i}\cdot e^k+1$, $y'(k+)=\sum_{i=0}^{k}e^{-i}\cdot e^k+1$.

Via Laplace transform, one obtains $s\cdot Y(s)-1-Y(s)=\mathcal{L}(f)(s)$ thus

$$Y(s)=\frac{1}{s-1}+\frac{1}{s-1}\cdot\mathcal{L}(f)(s).$$

So, the original Laplace is given by

$$\begin{aligned}
y(x) &= \mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\bigg(\frac{1}{s-1} \\
&\quad +\frac{1}{s-1}\cdot\sum_{k=0}^{n}\int_k^{k+1}e^{-sx}\cdot(-x+k+1)\,dx\bigg) \\
&= e^x \\
&\quad +\sum_{k=0}^{n}\mathcal{L}^{-1}\bigg[\bigg(-\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s-1}\bigg)\cdot e^{-(k+1)s} \\
&\quad +\frac{1}{s^2}\cdot e^{-ks}\bigg] \\
&= e^x+\sum_{k=0}^{n}\big[(-1-x+e^x)\cdot u(x-k-1) \\
&\quad +x\cdot u(x-k)\big]. \qquad (7)
\end{aligned}$$

The solution given in (6) *is a.e. with the one from* (7).

The next example traces the known line of the Laplace transform.

Example 3. *Let $I_k=]\frac{1}{k+1},\frac{1}{k}]$, $k\in\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. Let us consider the Cauchy problem*

$$(C_1)\quad \begin{cases} y'-y=f(x) \\ y(1)=0, \end{cases}$$

[5] Since $\frac{1}{s\cdot(s^2+\omega^2)}=\frac{1}{\omega^2}\cdot\left(\frac{1}{s}-\frac{s}{s^2+\omega^2}\right)$, from pg. 108 we get

$$\begin{aligned}
&\mathcal{L}(\cos\omega x)+i\cdot\mathcal{L}(\sin\omega x)=\mathcal{L}(e^{i\omega x}) \\
&= \int_0^\infty e^{-sx}\cdot e^{i\omega x}\,dx=\frac{1}{-s+i\omega}\cdot e^{-x(s-i\omega)}\big|_0^\infty \\
&= \frac{1}{s-i\cdot\omega}=\frac{s+i\cdot\omega}{s^2+\omega^2},
\end{aligned}$$

hence $\mathcal{L}(\cos\omega x)=\frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin\omega x)=\frac{\omega}{s^2+\omega^2}$.

where $f(x) = \frac{1}{k}$, $x \in \bigcup_{k\in\mathbb{N}^*} I_k \;=]0,1]$. *Since* $\int_{[0,1]} |f(x)|\,dx = \sum_{k=1}^{\infty} \frac{1}{k^2}\cdot\frac{1}{k+1} = \frac{\pi^2}{6} - 1 < +\infty$ *we have* $f \in L^1[0,1]$.

On $I_1 =]1/2,1]$ *we get the classic solution* $y_1(x) = -e^{-1}\cdot e^x - 1$. *We go on to solve the problem on* I_k, *successively for* $k \in \{2,3,...\}$

$$(C_{I_k})\quad \begin{cases} y' - y = f(x) \\ y(1/k) = y_{I_k}(1/k+), \end{cases}$$

finding the classic solution $y_{I_k}(x) = e^x \cdot \left[e^{-1} - \sum_{i=2}^{k} \frac{1}{i(i-1)}\cdot e^{-1/i}\right] - \frac{1}{k}$. *Then, the generalized solution of* (C_1) *is*

$$y(x) = e^x \cdot \left[e^{-1} - \sum_{i=1}^{\infty} \frac{1}{i(i+1)}\cdot e^{\frac{-1}{i+1}}\right],\ x \in [0,1], \tag{8}$$

with $x = 1/k$ *angular points* $y'(1/k-) = e^{1/k}\cdot\left[e^{-1} - \sum_{i=2}^{k}\frac{1}{i(i-1)}\cdot e^{-1/i}\right]$, $y'(1/k+) = e^{1/(k+1)}\cdot\left[e^{-1} - \sum_{i=2}^{k+1}\frac{1}{i(i-1)}\cdot e^{-1/i}\right]$.

The Laplace transform cannot be applied because of 2°, *the set of discontinuity points is not finite but it is Lebesgue negligible.*

3 Conclusions and future work

The previous example implies the role of Lebesgue integrability and Fourier transform.

Anyway, our point is that the Laplace transform, as a mathematical tool, should be introduced inductively instead of being defined before all.

We are going to implement the present approach to a computer algebra system.

References

[1] Bohner, M., Guseinov, G. Sh. (2010), The $h-$Laplace and $q-$Laplace transforms, J. Math. Anal. Appl. vol. 365, pp. 75-92.

[2] Brânzănescu, V., Stănăşilă, O. (1998), *Matematici speciale*, Editura All, Bucureşti (in Romanian).

[3] Cârtoaje, V. (2013), *Teoria Sistemelor Automate*, Editura Universităţii de Petrol-Gaze din Ploieşti, Ploieşti (in Romanian).

[4] Corduneanu, A. (1981), *Ecuaţii Diferenţiale cu Aplicaţii în Electrotehnică*, Editura Facla, Timişoara (in Romanian).

[5] Craddock, M., Lennox, K. A. (2007), Lie group symmetries as integral transforms of fundamental solutions, *J. Differential Equations*, vol. 232, pp. 652-674.

[6] Craddock, M., Lennox, K. A. (2009), Fundamental solutions, transition densities and the integration of Lie symmetries, *J. Differential Equations*, vol. 246, pp. 2538-2560.

[7] Datko, R. (1983), The Laplace transform and the integral stability of certain linear processes, *J. Differential Equations*, vol. 48, pp. 386-403.

[8] Davis, J. M., Gravagne, I. A., Jackson, B. J., Marks II, R. J., Ramos, A. A. (2007), The Laplace transform on time scales revisited, *J. Math. Anal. Appl.*, vol. 332, pp. 1291-1307.

[9] Dyke, P. (2014), *An Introduction to Laplace Transforms and Fourier Series*, Springer.

[10] Grote, K.-H., Antonsson, E. K. (2009), *Springer Handbook of Mechanical Engineering*, vol. 10, Springer.

[11] Kéyantuo, R. (1995), The Laplace transform and the ascent method for the abstract wave equation, *J. Differential Equations*, vol. 122, pp. 27-47.

[12] Micula, Gh., Pavel, P. (1989), *Ecuaţii diferenţiale şi integrale prin probleme şi exerciţii*, Editura Dacia, Cluj-Napoca (in Romanian).

[13] Pavel, G., Tomuţa, F. I., Gavrea, I. (1981), *Matematici speciale*, Editura Dacia, Cluj-Napoca (in Romanian).

[14] Redheffer, R. (1991), *Differential Equations: Theory and Applications*, Boston.

[15] Rezaei, H., Jung, S.-M., Rassias, T. M. (2013), Laplace transform and Hyers-Ulam stability of linear differential equations, *J. Math. Anal. Appl.*, vol. 403, pp. 244-251.