Scientific Bulletin of the „Petru Maior" University of Tîrgu Mureş
Vol. 15 (XXXII) no. 1, 2018
ISSN-L 1841-9267 (Print), ISSN 2285-438X (Online), ISSN 2286-3184 (CD-ROM)

REMARKS ON A GENERALIZATION OF A QUESTION RAISED BY PÁL ERDŐS CONCERNING A GEOMETRIC INEQUALITY IN ACUTE TRIANGLES II

Béla FINTA
"Petru Maior" University of Tîrgu Mureş
Nicolae Iorga Street, no.1, 540088, Tîrgu Mureş, Romania
e-mail: fintab@science.upm.ro

Abstract

The purpose of this paper is to give a negative answer to a possible generalization of an open question raised by Pál Erdős, concerning an inequality in acute triangles. We prove here that from $a < b < c$ does not follow $a^{2k+1} + (l_a)^{2k+1} < b^{2k+1} + l_b^{2k+1} < c^{2k+1} + l_c^{2k+1}$ in every acute triangle ABC, nor the opposite chain of inequalities, where $k \in \mathbb{N}, k \geq 2$, and a, b, c denotes the length of the triangles sites, while l_a, l_b, l_c denotes the length of the interior angle bisectors, as usual. We achieve this by constructing effectively two counterexamples, one for each type of inequalities.

Keywords: geometrical inequalities, acute triangle, interior bisectrices

1 Introduction

Let us consider the acute triangle ABC with sides $a = BC, b = AC$ and $c = AB$. In [1] appeared the following open question due to Pál Erdős: "if ABC is an acute triangle such that $a < b < c$ then $a + l_a < b + l_b < c + l_c$", where l_a, l_b, l_c means the length of the interior bisectrices corresponding to the sides BC, AC and AB, respectively.

In [2] Mihály Bencze proposed the following open question, which is a generalization of the problem of Pál Erdős: "determine all points $M \in Int(ABC)$, for which in case of $BC < CA < AB$ we have $CB + AA' < CA + BB' < AB + CC'$, where A',B',C' is the intersection of AM, BM, CM with sides BC,CA,AB". Here with $Int(ABC)$ we denote the interior points of the triangle ABC.

If we try for "usual" acute triangles ABC, we can verify the validity of the Erdős inequality. But in [3] we realized to obtain an acute triangle for which the Erdős inequality is false: let ABC be such that $c = 10 + \epsilon, b = 10$ and $a = 1$, where $\epsilon > 0$ is a "very small" positive quantity. Using the trigonometrical way combined with some elementary properties from algebra and mathematical analysis we showed that for this "extreme" acute triangle from $a < b < c$ results $c + l_c < b + l_b$.

In [4] József Sándor proved some new geometrical inequalities using in the open question of Pál Erdős instead of bisectrices altitudes and medians.

In [5] Károly Dáné and Csaba Ignát studied the open question of Mihály Bencze using the computer.

In connection with Erdős's problem we formulated the following open question: "if ABC is an acute triangle such that $a < b < c$ then $a^2 + l_a^2 < b^2 + l_b^2 < c^2 + l_c^2$". In [6] we proved the validity of this statement.

At the same time we formulated another new open question: "if ABC is an acute triangle such that $a < b < c$ then $a^4 + l_a^4 < b^4 + l_b^4 < c^4 + l_c^4$". In [7] we realized to find two acute triangles ABC such that in the first triangle from $a < b < c$ ($b = a + \epsilon, c = a + 2\epsilon$ with $\epsilon > 0$ a small positive quantity) we obtained $a^4 + l_a^4 < b^4 + l_b^4$, but in the second triangle from $a < b < c$ ($b = a + \epsilon, c = a\sqrt{2}$ with $\epsilon > 0$ a small positive quantity) we deduced $a^4 + l_a^4 > b^4 + l_b^4$. So the answer to our question is negative.

We can formulate the following more general open question using the problem of Pál Erdős: "if ABC is an acute triangle from $a < b < c$ results $a^\alpha + l_a^\alpha < b^\alpha + l_b^\alpha < c^\alpha + l_c^\alpha$, where $\alpha \in \mathbb{R}$. We mention, that this property is not obvious, because from $a < b < c$ results $l_a > l_b > l_c$. If $\alpha = 0$, then is immediately that our problem is false.

In [8] we proved for $\alpha = 2k \in \mathbb{N}, k \in \mathbb{N}, k \geq 2$ even natural number that the above inequality is not true in every acute triangle ABC, nor the opposite chain of inequalities.

2 Main part

The purpose of this paper is to study this open question for $\alpha = 2k+1, k \in \mathbb{N}$ odd natural number.

Proposition 1. *There exist acute triangles ABC with $a < b < c$, such that for these triangles we have $a^{2k+1} + (l_a)^{2k+1} < b^{2k+1} + (l_b)^{2k+1}$ for every odd natural number $2k+1, k \in \mathbb{N}$.*

Proof. We have the following sequence of equivalent inequalities: $a^{2k+1} + (l_a)^{2k+1} < b^{2k+1} + (l_b)^{2k+1} \Leftrightarrow (l_a)^{2k+1} - (l_b)^{2k+1} < b^{2k+1} - a^{2k+1} \Leftrightarrow (l_a - l_b) \cdot [(l_a)^{2k} + (l_a)^{2k-1} \cdot l_b + \cdots + l_a \cdot (l_b)^{2k-1} + (l_b)^{2k}] < (b-a) \cdot (b^{2k} + b^{2k-1} \cdot a + \cdots + b \cdot a^{2k-1} + a^{2k}) \Leftrightarrow \frac{(l_a)^2 - (l_b)^2}{l_a + l_b} \cdot [(l_a)^{2k} + (l_a)^{2k-1} \cdot l_b + \cdots + l_a \cdot (l_b)^{2k-1} + (l_b)^{2k}] < (b-a) \cdot (b^{2k} + b^{2k-1} \cdot a + \cdots + b \cdot a^{2k-1} + a^{2k})$.

But $l_a^2 = \frac{bc \cdot [(b+c)^2 - a^2]}{(b+c)^2}$ and $l_b^2 = \frac{ac \cdot [(a+c)^2 - b^2]}{(a+c)^2}$ so $l_a^2 - l_b^2 = bc - \frac{a^2bc}{(b+c)^2} - ac + \frac{ab^2c}{(a+c)^2} = c \cdot (b - a) + abc \cdot [\frac{b}{(a+c)^2} - \frac{a}{(b+c)^2}] = c \cdot (b-a) + abc \cdot \frac{(b-a) \cdot [(b^2 + ba + a^2) + 2c(b+a) + c^2]}{(a+c)^2 \cdot (b+c)^2} = (b-a) \cdot [c + abc \cdot \frac{b^2 + ba + a^2 + 2c(b+a) + c^2}{(a+c)^2 \cdot (b+c)^2}]$.

This means, that $a^{2k+1} + (l_a)^{2k+1} < b^{2k+1} + (l_b)^{2k+1} \Leftrightarrow \frac{1}{l_a + l_b} \cdot [c + abc \cdot \frac{b^2 + ba + a^2 + 2c(b+a) + c^2}{(a+c)^2 \cdot (b+c)^2}] \cdot [(l_a)^{2k} + (l_a)^{2k-1} \cdot l_b + \cdots + l_a \cdot (l_b)^{2k-1} + (l_b)^{2k}] < b^{2k} + b^{2k-1} \cdot a + \cdots + b \cdot a^{2k-1} + a^{2k}$.

At the first we choose the acute triangle ABC, such that $b = a + \epsilon$ and $c = a + 2\epsilon$, where $\epsilon > 0$ is a small positive quantity. Then $\lim_{\epsilon \to 0} (l_a)^2 = \lim_{\epsilon \to 0} \frac{bc \cdot [(b+c)^2 - a^2]}{(b+c)^2} = \lim_{\epsilon \to 0} \frac{(a+\epsilon)(a+2\epsilon) \cdot [(a+\epsilon+a+2\epsilon)^2 - a^2]}{(a+\epsilon+a+2\epsilon)^2} = \frac{3a^2}{4}$, $\lim_{\epsilon \to 0} l_a = \frac{\sqrt{3}a}{2}$,

$\lim_{\epsilon \to 0} (l_b)^2 = \lim_{\epsilon \to 0} \frac{ac \cdot [(a+c)^2 - b^2]}{(a+c)^2} = \lim_{\epsilon \to 0} \frac{a \cdot (a+2\epsilon) \cdot [(a+a+2\epsilon)^2 - (a+\epsilon)^2]}{(a+a+2\epsilon)^2} = \frac{3a^2}{4}$, $\lim_{\epsilon \to 0} l_b = \frac{\sqrt{3}a}{2}$, $\lim_{\epsilon \to 0} b = \lim_{\epsilon \to 0} (a + \epsilon) = a$, and $\lim_{\epsilon \to 0} [c + abc \cdot \frac{b^2 + ba + a^2 + 2c(b+a) + c^2}{(a+c)^2 \cdot (b+c)^2}] = \lim_{\epsilon \to 0} [(a + 2\epsilon) + a \cdot (a + \epsilon) \cdot (a + 2\epsilon) \cdot \frac{(a+\epsilon)^2 + (a+\epsilon) \cdot a + a^2 + 2 \cdot (a+2\epsilon) \cdot (a+\epsilon+a) + (a+2\epsilon)^2}{(a+a+2\epsilon)^2 \cdot (a+\epsilon+a+2\epsilon)^2}] = \frac{3a}{2}$.

This means, that $\lim_{\epsilon \to 0} \frac{1}{l_a + l_b} \cdot [c + abc \cdot \frac{b^2 + ba + a^2 + 2c(b+a) + c^2}{(a+c)^2 \cdot (b+c)^2}] \cdot [(l_a)^{2k} + (l_a)^{2k-1} \cdot l_b + \cdots + l_a \cdot (l_b)^{2k-1} + (l_b)^{2k}] \leq \lim_{\epsilon \to 0} (b^{2k} + b^{2k-1} \cdot a + \cdots + b \cdot a^{2k-1} + a^{2k})$, i.e. $\frac{1}{\frac{\sqrt{3}a}{2} + \frac{\sqrt{3}a}{2}} \cdot \frac{3a}{2} \cdot (\frac{\sqrt{3}a}{2})^{2k} \cdot (2k+1) \leq a^{2k} \cdot (2k+1)$, i.e. $(\frac{\sqrt{3}}{2})^{2k+1} \leq 1$.

We can see immediately that the inequality $(\frac{\sqrt{3}}{2})^{2k+1} < 1$ is true for all $k \in \mathbb{N}$. Using the definition of the limit we can conclude, that there exists a small positive real number $\epsilon_0 > 0$ such that for the triangle $A_0B_0C_0$ with $B_0C_0 = a, A_0C_0 = b = a + \epsilon_0$ and $A_0B_0 = c = a + 2\epsilon_0$ we obtain $a^{2k+1} + (l_a)^{2k+1} < b^{2k+1} + (l_b)^{2k+1}$ for every odd natural number $2k+1, k \in \mathbb{N}$. □

Proposition 2. *There exist acute triangles ABC with $a < b < c$, such that for these triangles we have $a^{2k+1} + (l_a)^{2k+1} > b^{2k+1} + (l_b)^{2k+1}$ for every odd natural number $2k+1, k \in \mathbb{N} - \{0\}$.*

Proof. Using the above presented sequence of ideas we get similarly, that $a^{2k+1} + (l_a)^{2k+1} > b^{2k+1} + (l_b)^{2k+1} \Leftrightarrow \frac{1}{l_a + l_b} \cdot [c + abc \cdot \frac{b^2 + ba + a^2 + 2c(b+a) + c^2}{(a+c)^2 \cdot (b+c)^2}] \cdot [(l_a)^{2k} + (l_a)^{2k-1} \cdot l_b + \cdots + l_a \cdot (l_b)^{2k-1} + (l_b)^{2k}] > b^{2k} + b^{2k-1} \cdot a + \cdots + b \cdot a^{2k-1} + a^{2k}$.

At the second we choose the acute triangle ABC, such that $b = a + \epsilon$ and $c = a\sqrt{2}$, where $\epsilon > 0$ is a small positive quantity. Then $\lim_{\epsilon \to 0} (l_a)^2 = \lim_{\epsilon \to 0} \frac{bc \cdot [(b+c)^2 - a^2]}{(b+c)^2} = \lim_{\epsilon \to 0} \frac{(a+\epsilon) \cdot a\sqrt{2} \cdot [(a+\epsilon+a\sqrt{2})^2 - a^2]}{(a+\epsilon+a\sqrt{2})^2} = (4 - 2\sqrt{2}) \cdot a^2$, $\lim_{\epsilon \to 0} l_a = \sqrt{4 - 2\sqrt{2}} \cdot a$,

$\lim_{\epsilon \to 0} (l_b)^2 = \lim_{\epsilon \to 0} \frac{ac \cdot [(a+c)^2 - b^2]}{(a+c)^2} = \lim_{\epsilon \to 0} \frac{a \cdot a\sqrt{2} \cdot [(a+a\sqrt{2})^2 - (a+\epsilon)^2]}{(a+a\sqrt{2})^2} = (4 - 2\sqrt{2}) \cdot a^2$, $\lim_{\epsilon \to 0} l_b = \sqrt{4 - 2\sqrt{2}} \cdot a$, $\lim_{\epsilon \to 0} b = \lim_{\epsilon \to 0} (a + \epsilon) = a$, and $\lim_{\epsilon \to 0} [c + abc \cdot \frac{b^2 + ba + a^2 + 2c(b+a) + c^2}{(a+c)^2 \cdot (b+c)^2}] = \lim_{\epsilon \to 0} [(a\sqrt{2}) + a \cdot (a + \epsilon) \cdot$

$(a\sqrt{2}) \cdot \frac{(a+\epsilon)^2+(a+\epsilon)\cdot a+a^2+2\cdot(a\sqrt{2})\cdot(a+\epsilon+a)+(a\sqrt{2})^2}{(a+a\sqrt{2})^2\cdot(a+\epsilon+a\sqrt{2})^2}] = (16-10\sqrt{2})\cdot a$.

This means, that $\lim_{\epsilon\to 0}\frac{1}{l_a+l_b} \cdot [c + abc \cdot \frac{b^2+ba+a^2+2c(b+a)+c^2}{(a+c)^2\cdot(b+c)^2}]\cdot[(l_a)^{2k}+(l_a)^{2k-1}\cdot l_b+\cdots+l_a\cdot(l_b)^{2k-1}+(l_b)^{2k}] \geq \lim_{\epsilon\to 0}(b^{2k}+b^{2k-1}\cdot a+\cdots+b\cdot a^{2k-1}+a^{2k})$, i.e. $\frac{1}{\sqrt{4-2\sqrt{2}}\cdot a+\sqrt{4-2\sqrt{2}}\cdot a}\cdot(16-10\sqrt{2})\cdot a\cdot(\sqrt{4-2\sqrt{2}}\cdot a)^{2k}\cdot(2k+1) \geq a^{2k}\cdot(2k+1)$, i.e. $(8-5\sqrt{2})\cdot(\sqrt{4-2\sqrt{2}})^{2k-1} \geq 1$. But for $k \geq 1$ we have $(8-5\sqrt{2})\cdot(\sqrt{4-2\sqrt{2}})^{2k-1} \geq (8-5\sqrt{2})\cdot\sqrt{4-2\sqrt{2}} > 1$.

So the inequality $(8-5\sqrt{2})\cdot(\sqrt{4-2\sqrt{2}})^{2k-1} > 1$ is true for all $k \in \mathbb{N}-\{0\}$. Using the definition of the limit we can conclude, that there exists a small positive real number $\epsilon_1 > 0$ such that for the triangle $A_1B_1C_1$ with $B_1C_1 = a, A_1C_1 = b = a+\epsilon_1$ and $A_1B_1 = c = a\sqrt{2}$ we obtain $a^{2k+1}+(l_a)^{2k+1} > b^{2k+1}+(l_b)^{2k+1}$ for every odd natural number $2k+1, k \in \mathbb{N}-\{0\}$. □

3 Discussion and conclusion

Conclusion Proposition 1 and 2 show that using the usual notations for the length of sides and the length of the interior angle bisectors of an acute triangle, if we suppose $a < b < c$ then does not result $a^{2k+1} + (l_a)^{2k+1} < b^{2k+1} + l_b^{2k+1} < c^{2k+1} + l_c^{2k+1}$, nor the opposite chain of the inequalities for any odd values greater than 1 of the exponents.

References

[1] Open Question OQ.14, Mathematical Magazine Octogon, Vol. 3, No. 1, 1995, pp. 54, Braşov, Romania

[2] Open Question OQ.27, Mathematical Magazine Octogon, Vol. 3, No. 2, 1995, pp. 64, Braşov, Romania

[3] Béla Finta, Solution for an Elementary Open Question of Pál Erdős, Mathematical Magazine Octogon, Vol. 4, No. 1, 1996, pp. 74-79, Braşov, Romania

[4] József Sándor, On Some New Geometric Inequalities, Mathematical Magazine Octogon, Vol. 5, No. 2, 1997, pp. 66-69, Braşov, Romania

[5] Károly Dáné, Csaba Ignát, On the Open Question of P. Erdős and M. Bencze, Mathematical Magazine Octogon, Vol. 6, No. 1, 1998, pp. 73-77, Braşov, Romania

[6] Béla Finta, A New Solved Question in Connection to a Problem of Pál Erdős, Proceedings of the 3rd Conference on the History of Mathematics and Teaching of Mathematics, University of Miskolc, May 21-23, 2004, pp. 56-60, Miskolc, Hungary

[7] Béla Finta, A New Solved Question in Connection to a Problem of Pál Erdős II, Didactica Matematicii, "Babeş-Bolyai" University, Vol. 24, 2006, pp. 65-70, Cluj-Napoca, Romania

[8] Béla Finta, Some Geometrical Inequalities in Acute Triangle, Lucrările celei de a III-a Conferinţe anuale a Societăţii de Ştiinţe Matematice din România, Vol. 3, Comunicări metodico-ştiinţifice, Universitatea din Craiova, 26-29 mai 1999, pp.193-200, Craiova, România

[9] Béla Finta, Remarks on a Generalisation of a Question Raised by Pál Erdős Concerning a Geometric Inequality in Acute Triangles, Scientific Bulletin of the "Petru Maior" University of Tirgu Mures, Vol. 14(XXXI), no. 2, pp. 33-35, ISSN-L 1841-9267 (Print), ISSN 2285-438X (Online), ISSN 2286-3184 (CD-ROM), 2017.