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COMPLETE SOLUTION FOR LINEAR PROGRAMMING PROBLEM IN TWO VARIABLES AND ONE RESTRICTION

Marcel Bogdan Bogdan-Emil Păcurar
University of Medicine, Pharmacy, Science and Technology Tg. Mureş, Romania
e-mail: marcel.bogdan@umfst.ro   pacurar.bogdan.emil@stud.upm.ro

ABSTRACT

In this work we consider an elementary linear programming problem with one restriction in order to be returned the set of solutions. For this, we include the corresponding source code.

Keywords: linear programming, simplex algorithm, multiple solutions.

1 Linear programming problem in two variables and one restriction

There are numerous applications that use linear programming [1, 3, 4, 5, 6, 7, 8, 9]. In general, a command from a computer algebra system finds a vector that minimizes a linear objective function subject to the constraints. For example, by using WolframAlpha [10], $LinearProgramming[c, a, b]$ finds a vector x (one vector) that minimizes $\langle c, x\rangle$ subject to the constraints $a \cdot x \geq b$, $x \geq 0$. By solving the linear problem formulated we are looking to return the set of solutions. As it is formulated in the title, we consider the following basic linear programming problem

$$\begin{cases} c_1 \cdot x_1 + c_2 \cdot x_2 \longrightarrow \min \\ a_1 \cdot x_1 + a_2 \cdot x_2 \leq b \\ x_1, x_2 \geq 0, \end{cases} \tag{1}$$

where $(c_1, c_2) = c \in \mathbb{R}^2$, $(a_1, a_2) = a \in \mathbb{R}^2$, and $b \in \mathbb{R}$ are given. Denote by $A = \mathbb{R}^2_+ \cap H^{\leq}$, the admissible set, where

$$H^{\leq} = \{(x_1, x_2) \in \mathbb{R}^2 \,|a_1 x_1 + a_2 x_2 \leq b\}$$

is the usual halfplane. If A is nonempty and bounded, then a solution exists for the problem $\min_{(x_1,x_2)\in A} f(x_1, x_2)$, since $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = c_1 \cdot x_1 + c_2 \cdot x_2$ is continuous.

Remark 1. *If $c_1 = c_2 = 0$, the set of solutions equals A.*

If $a_1 = a_2 = 0$, $c_1^2 + c_2^2 > 0$, and $b \geq 0$, the solution is $(0, 0)$ if $c_1, c_2 > 0$, the solution is $[0, +\infty) \times \{0\}$ if $c_1 = 0$, $c_2 > 0$, the solution is $\{0\} \times [0, +\infty)$ if $c_1 > 0$, $c_2 = 0$, otherwise there is no solution. If $b < 0$, there is no solution since $A = \emptyset$.

Suppose further on, that $c_1^2 + c_2^2 > 0$ and $a_1^2 + a_2^2 > 0$. To find a solution, if exists, it is elementary (geometrically or other way, including CASystems [10, 11]). Note that CASystems provide at most one solution, regardless the implemented algorithm. For particular given data c, a, and b, to completely solve the problem, it is not difficult. Precisely, we are going to use the extended simplex algorithm, recently proposed in [2], motivated by the criteria for the nonuniqueness of the solution. Mainly, it consists on deciding when the set of solutions is a singleton, a convex hull of a finite number of solutions, or an unbounded set (see Proposition 1). Anyway, to close the problem, to cover it all, is our task. Since, there should be a start for the commonly problem in $\mathbb{R}^n$ (at least $n = 3$ is achievable), up to several restrictions, not exclusively.

If $a_1 = 0$, then $\partial H^{\leq} \cap Ox_1 = \emptyset$; else, let $\partial H^{\leq} \cap Ox_1 = \{\bar{x}_0\} = \{(b/a_1, 0)\}$. If $a_2 = 0$, then $\partial H^{\leq} \cap Ox_2 = \emptyset$; else, let $\partial H^{\leq} \cap Ox_2 = \{\bar{x}_0'\} = \{(0, b/a_2)\}$.

Let us consider the following two cases:

I. $(0, 0) \in H^{\leq}$, that is $b \geq 0$;

II. $(0, 0) \notin H^{\leq}$, that is $b < 0$.

For each case we distinguish three situations that lead to the nontrivial cases of the admissible set, of an itself triangle, respectively of an unbounded set; we prefer to list them like this: *i)* $\bar{x}_0, \bar{x}'_0 \in A$; *ii)* $\bar{x}_0 \in A, \bar{x}'_0 \notin A$; *iii)* $\bar{x}'_0 \in A, \bar{x}_0 \notin A$.

Remark 2. *The remaining situation $\bar{x}_0, \bar{x}'_0 \notin A$, has a trivial conclusion, if $b \geq 0$, then $A = \mathbb{R}^2_+$, therefore $(0,0)$ is the unique solution for $c_1, c_2 > 0$; there is no solution if $c_1 < 0$ or $c_2 < 0$; the set of solutions is $[0,+\infty) \times \{0\}$ if $c_1 = 0, c_2 > 0$ and $\{0\} \times [0,+\infty)$ if $c_1 > 0, c_2 = 0$. If $b < 0$, then $A = \emptyset$ thus no solution exists.*

To initiate the algorithm, the standard form is
$$\begin{cases} c_1x_1 + c_2x_2 \longrightarrow \min \\ a_1x_1 + a_2x_2 + x_3 = b \quad \text{with data } c = (c_1, c_2, 0) \\ x_1, x_2, x_3 \geq 0, \end{cases}$$
and $(a_1\ a_2\ 1) = (A^j)_{1\leq j\leq 3}$. The usual notations are the following: $b = \alpha_{0j}A^j$, (in this particular case $\mathcal{B} = \{j\}$ if $a_j \geq 0$ or $\mathcal{B} = \{3\}$ if $b = A^3 = 1$ is chosen), $A^i = \alpha_{ij}A^j$, and $\alpha_{i0} = \alpha_{ij}c_j - c_i$, for $i \in \overline{\mathcal{B}} = \{1,2,3\} \setminus \mathcal{B}$.

Let us denote by $\mathcal{S}_{st}$ the set of solutions and by $\mathcal{S}$ the set of solutions for (1). In order to be completely determined we refer to the following statement (see [2]).

Proposition 1. *Suppose that B is the optimal basis and x^B is a solution obtained by the simplex algorithm. The following implications apply:*

1. *If $\max_{i\in\overline{\mathcal{B}}} \alpha_{i0} < 0$, then $\mathcal{S}_{st} = \{x^B\}$.*
2. *If $\max_{i\in\overline{\mathcal{B}}} \alpha_{i0} = 0$ and $\min_{j\in\mathcal{B}} \alpha_{0j} = 0$, then $\mathcal{S}_{st} = \{x^B\}$.*
3. *If $\max_{i\in\overline{\mathcal{B}}} \alpha_{i0} = 0$ and $\min_{j\in\mathcal{B}} \alpha_{0j} > 0$, then $\{x^B\} \subsetneq \mathcal{S}_{st}$. If more:*
   - *α) for some $\bar{i} \in \overline{\mathcal{B}}_0 = \{i \in \overline{\mathcal{B}} \,|\, \alpha_{i0} = 0\}$, there exists $k \in \mathcal{B}$ such that $\alpha_{\bar{i}k} > 0$, then $\mathcal{S}_{st} = co\{x^B, x^{B'}\}$, where $x^{B'} \neq x^B$;*
   - *β) for some $\bar{i} \in \overline{\mathcal{B}}_0$, $\alpha_{\bar{i}k} \leq 0, \forall k \in \mathcal{B}$, then*
   $$\mathcal{S}_{st} = \{x^B + \alpha\cdot\bar{c} \,|\, \alpha \geq 0\}, \ \ \bar{c}^T = (\bar{c}_1, \bar{c}_2, \bar{c}_3),$$
   where the components are defined by $\bar{c}_j = \begin{cases} -\alpha_{\bar{i}j}, & j \in \mathcal{B} \\ 1, & j = \bar{i} \\ 0, & \textit{otherwise}. \end{cases}$

2 Results

Let us denote by $d = \begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix}$. The following two basic propositions distinguish the solutions while imposing the hypothesis $d \neq 0$ or $d = 0$. Although are basic results along with their proofs, these are given for the completion.

Proposition 2. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ be such that $d \neq 0$.*

Then, the following assertions apply:

I. i) the solution $x_0 \in A$ is such that
$$f(x_0) = \min\{f(0,0), f(\bar{x}_0), f(\bar{x}'_0)\}.$$
$$\textit{More } x_0 = \begin{cases} (0,0), & \text{if } c_1, c_2 > 0 \\ \bar{x}'_0, & \text{if } c_1 > 0, c_2 < 0 \\ \bar{x}_0, & \text{if } c_1 < 0, c_2 > 0 \\ \bar{x}'_0, & \text{if } c_1 < 0, c_2 < 0, d > 0 \\ \bar{x}_0, & \text{if } c_1 < 0, c_2 < 0, d < 0. \end{cases}$$

ii) if $c_2 < 0$, then $\mathcal{S} = \emptyset$;
if $c_2 > 0$, $c_1 < 0$, and $d > 0$, then $\mathcal{S} = \emptyset$;
if $c_2 > 0$, then $\mathcal{S} = \{x_0\}$, where x_0 is such that
$$\begin{aligned} f(x_0) &= \min\{f(0,0), f(\bar{x}_0)\} \\ &= \begin{cases} f(0,0), & c_1 > 0 \\ f(\bar{x}_0), & c_1 < 0, d < 0; \end{cases} \end{aligned}$$

iii) if $c_1 < 0$, then $\mathcal{S} = \emptyset$;
if $c_1 > 0$, $c_2 < 0$, and $d < 0$, then $\mathcal{S} = \emptyset$;
if $c_1 > 0$, then $\mathcal{S} = \{x_0\}$, where x_0 is such that
$$\begin{aligned} f(x_0) &= \min\{f(0,0), f(\bar{x}'_0)\} \\ &= \begin{cases} f(0,0), & c_2 > 0 \\ f(\bar{x}'_0), & c_2 < 0, d > 0. \end{cases} \end{aligned}$$

II. i) if $c_1 < 0 \vee c_2 < 0$, then $\mathcal{S} = \emptyset$;
if $c_1 > 0$ and $c_2 > 0$, then $\mathcal{S} = \begin{cases} \{\bar{x}_0\}, & \text{if } d > 0 \\ \{\bar{x}'_0\}, & \text{if } d < 0; \end{cases}$

ii) if $c_1 < 0$, then $\mathcal{S} = \emptyset$;
$$\textit{if } c_1 > 0, \textit{ then } \mathcal{S} = \begin{cases} \emptyset, & d < 0 \\ \{\bar{x}_0\}, & d > 0; \end{cases}$$

iii) if $c_2 < 0$, then $\mathcal{S} = \emptyset$;
$$\textit{if } c_2 > 0, \textit{ then } \mathcal{S} = \begin{cases} \emptyset, & d > 0 \\ \{\bar{x}'_0\}, & d < 0. \end{cases}$$

Proof. For all the implications we use tables $T1$, $T2$, and $T3$ from figures 1 and 2, respectively.

I. This case considers $b \geq 0$. Let $c_1, c_2 > 0$. By $T1$, we have $B = \{A^3\}$ optimal, therefore the solution is $(0,0,b)$, thus $x_0 = (0,0)$. The solution is unique by Proposition 1, since $\max\{\alpha_{10}, \alpha_{20}\} < 0$.

i) This situation is for $a_1, a_2 > 0$. Let $c_1 > 0$ and $c_2 < 0$, therefore $d > 0$. By $T2$ one obtains $B = \{A^2\}$ optimal basis so $x_0 = (0, b/a_2) = \bar{x}'_0$. Let $c_1 < 0$ and $c_2 > 0$, thus $d < 0$. By $T3$, $B = \{A^1\}$ is optimal, therefore $x_0 = (b/a_1) = \bar{x}_0$.

Let $c_1 < 0$ and $c_2 < 0$. By $T2$ and $T3$, respectively,
$$x_0 = \begin{cases} \bar{x}'_0, & \text{if } d > 0 \\ \bar{x}_0, & \text{if } d < 0. \end{cases}$$

	A^3	
A^1	a_1	$-c_1$
A^2	a_2	$-c_2$
	b	

	A^2	
A^1	a_1/a_2	$-d/a_2$
A^3	$1/a_2$	c_2/a_2
	b/a_2	

Figure 1. Tabels $T1$ and $T2$

	A^1	
A^3	$1/a_1$	c_1/a_1
A^2	a_2/a_1	d/a_1
	b/a_1	

Figure 2. Tabel $T3$

ii) Here we have $a_1 > 0$ and $a_2 < 0$. We have to prove that there is no solution if $c_2 < 0$. The same for $c_1 < 0$, $c_2 > 0$, and $d > 0$.

Let $c_2 < 0$. By $T1$, the algorithm stops since $\alpha_{20} = -c_2 > 0$ and $\alpha_{23} = a_2 < 0$, so no solution exists.

Let $c_1 < 0$, $c_2 > 0$. By $T3$, $x_0 = \bar{x}_0$ for $d < 0$, while for $d > 0$ there is no solution since $\alpha_{20} = d/a_1 > 0$ and $\alpha_{21} = a_2/a_1 < 0$.

iii) For this situation $a_1 < 0$ and $a_2 > 0$. Let $c_1 < 0$. From $T1$, since $\alpha_{10} = -c_1 > 0$ and $\alpha_{13} = a_1 < 0$, there is no solution.

Let $c_1 > 0$ and $c_2 < 0$. By $T2$, $x_0 = \bar{x}'_0$ for $d > 0$, while for $d < 0$ there is no solution since $\alpha_{10} = -d/a_2 > 0$ and $\alpha_{12} = a_1/a_2 < 0$.

II. i) For this situation $b < 0$, $a_1, a_2 < 0$.

In this case, we prove that if $c_1 < 0 \vee c_2 < 0$, then $\mathcal{S} = \emptyset$. Indeed, for $c_1 < 0$, see $T3$ with $B = \{A^1\}$ primal basis, $\alpha_{30} = c_1/a_1 > 0$ and $\alpha_{31} = 1/a_1 < 0$; for $c_2 < 0$, see $T2$ with $B = \{A^2\}$ primal basis, $\alpha_{30} = c_2/a_1 > 0$ and $\alpha_{32} = 1/a_2 < 0$.

Let $c_1 > 0$ and $c_2 > 0$. If $d > 0$, by $T3$, $B = \{A^1\}$ is optimal, so $x_0 = \bar{x}_0$ is solution. If $d < 0$, by $T2$, $B = \{A^2\}$ is optimal, therefore $x_0 = \bar{x}'_0$ is solution.

ii) In this case $b < 0$, $a_1 < 0$, and $a_2 > 0$.

Let $c_1 < 0$. By $T3$, $B = \{A^1\}$ is primal admissible, $\alpha_{30} = c_1/a_1 > 0$ and $\alpha_{31} = 1/a_1 < 0$, so $\mathcal{S} = \emptyset$.

Let $c_1 < 0$. By $T3$, if $d > 0$, $B = \{A^1\}$ is optimal, thus $x_0 = \bar{x}_0$ is solution. If $d < 0$, there is no solution since $B = \{A^1\}$ is primal admissible, $\alpha_{20} = d/a_1 > 0$ and $\alpha_{21} = a_2/a_1 < 0$, so $\mathcal{S} = \emptyset$.

iii) For this case $b < 0$, $a_1 > 0$, and $a_2 < 0$. By $T2$, $B = \{A^2\}$ is primal admissible. Let $c_2 < 0$. Since $\alpha_{30} = c_2/a_2 > 0$ and $\alpha_{32} = 1/a_2 < 0$, there is no solution. Let $c_2 < 0$. If $d < 0$, $B = \{A^2\}$ is optimal, therefore $x_0 = \bar{x}'_0$ is solution, otherwise there is no solution. □

Corollary 1. *If $d \neq 0$, then* (1) *has at most one solution.*

Proposition 3. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ be such that $d = 0$.*

Then, the following assertions apply:

I. i) if $c_1, c_2 > 0$, then $\mathcal{S} = \{(0, 0)\}$;
if $c_1, c_2 < 0$, then $\mathcal{S} = [\bar{x}_0, \bar{x}'_0]$;

ii) if $c_1 > 0$ and $c_2 < 0$, then $\mathcal{S} = \emptyset$;
if $c_1 < 0$, $c_2 > 0$, then

$$\mathcal{S} = \{\bar{x}_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\},\ \textit{where}\ \bar{c} = (-a_2/a_1, 1);$$

iii) if $c_1 < 0$ and $c_2 > 0$, then $\mathcal{S} = \emptyset$;
if $c_1 > 0$ and $c_2 < 0$, then

$$\mathcal{S} = \{\bar{x}'_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\},\ \textit{where}\ \bar{c} = (-a_1/a_2, 1);$$

II. i) if $c_1, c_2 > 0$, then $\mathcal{S} = [\bar{x}_0, \bar{x}'_0]$;
if $c_1, c_2 < 0$, then $\mathcal{S} = \emptyset$;

ii) if $c_1 > 0$, $c_2 < 0$, then

$$\mathcal{S} = \{\bar{x}_0 + \alpha \cdot (-a_2/a_1, 1) \,|\, \alpha \geq 0\};$$

if $c_1 < 0$ and $c_2 > 0$, then $\mathcal{S} = \emptyset$;

iii) if $c_1 > 0$ and $c_2 < 0$, then $\mathcal{S} = \emptyset$;
if $c_1 < 0$ and $c_2 > 0$, then

$$\mathcal{S} = \{\bar{x}'_0 + \alpha \cdot (-a_1/a_2, 1) \,|\, \alpha \geq 0\}.$$

Proof.

I. i) Recall that $b \geq 0$ and $a_1, a_2 > 0$. Let $c_1, c_2 > 0$. By $T1$ we get solution $(0, 0)$, that is unique since $\max\{\alpha_{10}, \alpha_{20}\} < 0$.

Let $c_1, c_2 < 0$. By $T2$ and $T3$, $B = \{A^2\}$ and $B' = \{A^1\}$ are optimal, respectively, we obtain $\mathcal{S} = [\bar{x}_0, \bar{x}'_0]$.

ii) In this case $b \geq 0$, $a_1 > 0$, and $a_2 < 0$. Let $c_1 > 0$ and $c_2 < 0$. By $T1$, $B = \{A^3\}$ is primal admissible, $\mathcal{S} = \emptyset$ because $\alpha_{20} = -c_2 > 0$ and $\alpha_{23} = a_2 < 0$.

Let $c_1 < 0$, $c_2 > 0$. By $T3$, $\alpha_{20} = 0$, so $\overline{\mathcal{B}}_0 = \{2\}$. From Proposition 1 we obtain that $\mathcal{S}_{st}$ is unbounded and

$$\mathcal{S}_{st} = \{(\bar{x}_0, 0) + \alpha \cdot (-\alpha_{21}, 1, 0) \,|\, \alpha \geq 0\},$$

hence

$$\mathcal{S} = \{\bar{x}_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\},$$

where $\bar{c} = (-\alpha_{21}, 1) = (-a_2/a_1, 1)$.

iii) Here $b \geq 0$, $a_1 < 0$, and $a_2 > 0$. Let $c_1 < 0$ and $c_2 > 0$. By $T1$, since $\alpha_{10} = -c_1 > 0$ and $\alpha_{13} = a_1 < 0$, it follows $\mathcal{S} = \emptyset$.

Let $c_1 > 0$ and $c_2 < 0$. By $T2$, $\alpha_{10} = 0$, so $\overline{\mathcal{B}}_0 = \{1\}$, therefore one obtains

$$\mathcal{S} = \{\bar{x}'_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\},$$

where $\bar{c} = (-\alpha_{12}, 1) = (-a_1/a_2, 1)$.

II. i) Let $c_1, c_2 > 0$, then by $T2$ and $T3$ we get $\mathcal{S} = [\bar{x}_0, \bar{x}'_0]$. Let $c_1, c_2 < 0$. By $T2$, $B = \{A^2\}$ is primal admissible, $\alpha_{30} = c_2/a_2 > 0$ and $\alpha_{32} = 1/a_2 < 0$, therefore $\mathcal{S} = \emptyset$.

ii) If $c_1 > 0$, $c_2 < 0$, then by $T3$, $\alpha_{20} = 0$, so $\overline{\mathcal{B}}_0 = \{2\}$, therefore we obtain

$$\mathcal{S} = \{\bar{x}_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\},$$

where $\bar{c} = (-\alpha_{21}, 1) = (-a_2/a_1, 1)$.

Let $c_1 < 0$ and $c_2 > 0$. By $T3$, because $\alpha_{30} = c_1/a_1 > 0$ and $\alpha_{31} = 1/a_1 < 0$, the conclusion follows $\mathcal{S} = \emptyset$.

iii) Here $b < 0$, $a_1 > 0$, and $a_2 < 0$. Let $c_1 > 0$ and $c_2 < 0$. By $T2$, because $\alpha_{30} = c_2/a_2 > 0$ and $\alpha_{32} = 1/a_2 < 0$, again $\mathcal{S} = \emptyset$.

Let $c_1 < 0$, $c_2 > 0$. By $T2$, $\alpha_{10} = 0$, so $\overline{\mathcal{B}}_0 = \{1\}$, therefore we obtain

$$\mathcal{S} = \{\bar{x}'_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\},$$

where $\bar{c} = (-\alpha_{12}, 1) = (-a_1/a_2, 1)$. □

From the previous two propositions one can recover the necessary condition for $\mathcal{S}$ not to be bounded, that is $d = 0$ (see [2], Proposition 4.5). This comes from the condition for the gradients of f and $\partial H^{\leq}$ to be linearly dependent, i.e. $\exists u \in \mathbb{R}$, $u \neq 0$ such that $\nabla f = (c_1, c_2) = u \cdot \nabla \partial H^{\leq} = u \cdot (a_1, a_2)$. More, $\mathcal{S}$ is unbounded only if $c_1 \cdot c_2 \leq 0$.

Structurally, by the data a, b, c, we spread the following items:

A. $c_1^2 + c_2^2 > 0$ and $a_1^2 + a_2^2 > 0$;

$A1$. $c_1 \cdot c_2 \neq 0$, $a_1 \cdot a_2 \neq 0$;

$A11$. $d \neq 0$, and the cases

I. $(0,0) \in H^{\leq}$, $b \geq 0$; $i)$ $\bar{x}_0, \bar{x}'_0 \in A$; $ii)$ $\bar{x}_0 \in A, \bar{x}'_0 \notin A$; $iii)$ $\bar{x}'_0 \in A, \bar{x}_0 \notin A$.

II. $(0,0) \notin H^{\leq}$, $b < 0$; $i)$, $ii)$, $iii)$ in Proposition 2.

$A12$. $d = 0$ along with the cases I. $i)$, $ii)$, $iii)$; II. $i)$, $ii)$, $iii)$ in Proposition 3.

$A2$. $c_1 \cdot c_2 = 0$; $a_1 \cdot a_2 \neq 0$; $I. - II.$, $i) - iii)$; (Proposition 4, Proposition 5)

$A3$. $a_1 \cdot a_2 = 0$; $c_1 \cdot c_2 \neq 0$;

$A31$. $I. - II.$, $i') - ii')$; $a_1 = 0$, $a_2 \neq 0$; (Proposition 6)

$A32$. $I. - II.$ $i'') - ii'')$; $a_1 \neq 0$, $a_2 = 0$; (Proposition 7).

B. $c_1 = c_2 = 0$; $a_1 = a_2 = 0$; in Remark 1

$c_2 = 0$, $a_1 = 0$; (Proposition 8)

$c_1 = 0$, $a_2 = 0$; (Proposition 9).

For item $A2$ we assign the next four propositions.

Proposition 4. *Let $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 = 0$, and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$.*

Then, the following assertions apply:

I. i) *if $c_1 > 0$, then $\mathcal{S} = [(0,0), \bar{x}'_0]$;*
if $c_1 < 0$, then $\mathcal{S} = \{\bar{x}_0\}$;

ii) *if $c_1 > 0$, then $\mathcal{S} = \{0\} \times [0, +\infty)$;*
if $c_1 < 0$, then $\mathcal{S} = \emptyset$;

iii) *if $c_1 > 0$, then $\mathcal{S} = [(0,0), \bar{x}'_0]$;*
if $c_1 < 0$, then $\mathcal{S} = \emptyset$;

II. i) $$\mathcal{S} = \begin{cases} \{\bar{x}'_0 + \alpha \cdot (0, -1/a_2) \,|\, \alpha \geq 0\}, & \text{if } c_1 > 0 \\ \emptyset, & \text{if } c_1 < 0; \end{cases}$$

ii) $$\mathcal{S} = \begin{cases} \{\bar{x}_0\}, & \text{if } c_1 > 0 \\ \emptyset, & \text{if } c_1 < 0; \end{cases}$$

iii) $$\mathcal{S} = \begin{cases} \{\bar{x}'_0 + \alpha \cdot (0, -1/a_2) \,|\, \alpha \geq 0\}, & \text{if } c_1 > 0 \\ \emptyset, & \text{if } c_1 < 0. \end{cases}$$

Proof. In this setting $d = c_1 \cdot a_2 \neq 0$.

I. i) This case considers $b \geq 0$ and $a_1, a_2 > 0$. First, let $b > 0$. Let $c_1 > 0$, thus $d > 0$. By $T1$, we have $B = \{A^3\}$ optimal with $\alpha_{20} = 0$, $\overline{\mathcal{B}}_0 = \{2\}$, by $T2$, $B = \{A^2\}$ is optimal with $\alpha_{30} = 0$, $\overline{\mathcal{B}}_0 = \{3\}$, therefore the solution is $\mathcal{S} = co\{(0,0), \bar{x}'_0\}$. If $b = 0$, then $\mathcal{S}$ reduces to $\{(0,0)\}$.

Let $c_1 < 0$, thus $d < 0$. By $T3$, $B = \{A^1\}$ is optimal and $\mathcal{S} = \{\bar{x}_0\}$. The solution is unique since $\max\{\alpha_{20}, \alpha_{30}\} < 0$.

ii) This case corresponds to $b \geq 0$, $a_1 > 0$, and $a_2 < 0$.

Let $c_1 > 0$. By $T1$, $B = \{A^3\}$ is optimal with $\alpha_{20} = -c_2 = 0$, $\alpha_{23} = a_2 < 0$, $\mathcal{B} = \{3\}$. By Proposition 1 $3\beta)$, with $\bar{i} = 2$, $\mathcal{S}$ is unbounded and

$$\mathcal{S}_{st} = \{(0,0,b) + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\}, \ \bar{c} = (0, 1, -\alpha_{23})$$

therefore

$$\mathcal{S} = \{(0,0) + \alpha \cdot (0,1) \,|\, \alpha \geq 0\} = \{(0,\alpha) \,|\, \alpha \geq 0\}.$$

Let $c_1 < 0$, so $d > 0$. By $T3$, there is no solution since $\alpha_{20} = d/a_1 > 0$ and $\alpha_{21} = a_2/a_1 < 0$.

iii) We have $b \geq 0$, $a_1 < 0$, and $a_2 > 0$. Let $c_1 > 0$. From $T1$, $B = \{A^3\}$ is optimal with $\alpha_{20} = 0$. From $T2$, $B = \{A^2\}$ is optimal with $\alpha_{30} = 0$. Once again, Proposition 1 3.α) concludes that $\mathcal{S} = [(0,0), \bar{x}'_0]$, unless $b \neq 0$; for $b = 0$ it becomes a singleton.

Let $c_1 < 0$. By $T1$, ($T3$ can be used as well) there is no solution since $\alpha_{10} = -c_1 > 0$ and $\alpha_{13} = a_1 < 0$.

II. i) This case considers $b < 0$, $a_1 < 0$, and $a_2 < 0$. Let $c_1 > 0$, thus $d < 0$. By $T2$, $B = \{A^2\}$ is optimal since $\alpha_{20} = -d/a_2 < 0$ and $\alpha_{30} = c_2/a_2 = 0$. Also, $\alpha_{32} = 1/a_2 < 0$, therefore, by Proposition 1 3.β), $\mathcal{S}$ is unbounded and $\mathcal{S} = \{\bar{x}'_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\}$, $\bar{c} = (0, -1/a_2)$.

Let $c_1 < 0$. By $T3$, $B = \{A^1\}$ is primal admissible, $\alpha_{30} = c_1/a_1 > 0$ and $\alpha_{31} = 1/a_1 < 0$, therefore $\mathcal{S} = \emptyset$.

ii) Here $b < 0$, $a_1 < 0$, and $a_2 > 0$. Let $c_1 > 0$, so $d > 0$. Since $\alpha_{01} = b/a_1 > 0$, $\alpha_{20} = d/a_1 < 0$, and $\alpha_{30} = c_1/a_1 < 0$, by $T3$, $B = \{A^1\}$ is optimal, therefore $\mathcal{S} = \{\bar{x}_0\}$.

Let $c_1 < 0$. From $T3$, $B = \{A^1\}$ is primal admissible. Since $\alpha_{30} = c_1/a_1 > 0$ and $\alpha_{31} = 1/a_1 < 0$, $\mathcal{S} = \emptyset$.

iii) In this case $b < 0$, $a_1 > 0$, and $a_2 < 0$. Let $c_1 > 0$. By $T2$, $B = \{A^2\}$ is optimal ($\mathcal{B}_0 = \{2\}$) since $\alpha_{02} = b/a_2 > 0$, $\alpha_{30} = c_2/a_2 = 0$, and $\alpha_{32} = 1/a_2 < 0$, therefore $\bar{x}'_0$ is a solution. By Proposition 1, $\mathcal{S} = \{\bar{x}'_0 + \alpha \cdot \bar{c} \,|\, \alpha \geq 0\}$, where $\bar{c} = (0, -\alpha_{32}) = (0, -1/a_2)$.

Let $c_1 < 0$. By $T2$, $\alpha_{10} = -d/a_2 > 0$, $\alpha_{12} = a_1/a_2 < 0$, so $\mathcal{S} = \emptyset$. ☐

Proposition 5. *Let $c_1 = 0$, $c_2 \in \mathbb{R} \setminus \{0\}$, and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$.*

Then, the following assertions apply:

I. i) if $c_2 > 0$, *then* $\mathcal{S} = [(0,0), \bar{x}_0]$;
if $c_2 < 0$, *then* $\mathcal{S} = \{\bar{x}'_0\}$;

ii) if $c_2 > 0$, *then* $\mathcal{S} = [(0,0), \bar{x}_0]$;
if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$;

iii) if $c_2 > 0$, *then* $\mathcal{S} = [0, +\infty) \times \{0\}$;
if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$;

II. i) if $c_2 > 0$, *then* $\mathcal{S} = [b/a_1, +\infty) \times \{0\} = \{\bar{x}_0 + (\alpha, 0) \,|\, \alpha \geq 0\}$;
if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$;

ii) if $c_2 > 0$, *then* $\mathcal{S} = [b/a_1, +\infty) \times \{0\} = \{\bar{x}_0 + (\alpha, 0) \,|\, \alpha \geq 0\}$;
if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$;

iii) if $c_2 > 0$, *then* $\mathcal{S} = \{\bar{x}'_0\}$;
if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$.

Proof. In this setting $d = -c_2 \cdot a_1 \neq 0$. The proof is similar to the previous one therefore it is omitted. ☐

If $a_1 \cdot a_2 = 0$ we assert the following.

Proposition 6. *Let $a_1 = 0$, $a_2 \neq 0$. Then, the following hold:*

I. i') $b \geq 0$, $a_2 > 0$.
If $c_1, c_2 > 0$, *then* $\mathcal{S} = \{(0,0)\}$;
if $c_1 < 0$, *then* $\mathcal{S} = \emptyset$;
if $c_1 > 0, c_2 < 0$, *then* $\mathcal{S} = \{\bar{x}'_0\}$;

ii') $b \geq 0$, $a_2 < 0$.
We have

$$\mathcal{S} = \begin{cases} \{(0,0)\}, & c_1 > 0, c_2 > 0 \\ [0, +\infty) \times \{0\}, & c_1 < 0, c_2 > 0 \\ \{0\} \times [0, +\infty), & c_1 > 0, c_2 < 0 \\ \emptyset, & c_1 < 0 \vee c_2 < 0; \end{cases}$$

II. i') $b < 0$, $a_2 > 0$; $A = \emptyset$, $\mathcal{S} = \emptyset$.

ii') $b < 0$, $a_2 < 0$.
We have

$$\mathcal{S} = \begin{cases} \{\bar{x}'_0\}, & c_1 > 0, c_2 > 0 \\ \emptyset, & c_1 < 0 \vee c_2 < 0. \end{cases}$$

Proposition 7. *Let $a_1 \neq 0$, $a_2 = 0$. Then, the following hold:*

I. i") $b \geq 0$, $a_1 > 0$.
If $c_1, c_2 > 0$, *then* $\mathcal{S} = \{(0,0)\}$;
if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$;
if $c_1 < 0, c_2 > 0$, *then* $\mathcal{S} = \{\bar{x}_0\}$;

ii") $b \geq 0$, $a_1 < 0$. $A = \mathbb{R}^2_+$.
One has

$$\mathcal{S} = \begin{cases} \{(0,0)\}, & c_1 > 0, c_2 > 0 \\ [0, +\infty) \times \{0\}, & c_1 < 0, c_2 > 0 \\ \{0\} \times [0, +\infty), & c_1 > 0, c_2 < 0 \\ \emptyset, & c_1 < 0 \vee c_2 < 0; \end{cases}$$

II. i") $b < 0$, $a_1 > 0$; $A = \emptyset$, $\mathcal{S} = \emptyset$.

ii") $b < 0$, $a_1 < 0$.
One has

$$\mathcal{S} = \begin{cases} \{\bar{x}_0\}, & c_1 > 0, c_2 > 0 \\ \emptyset, & c_1 < 0 \vee c_2 < 0. \end{cases}$$

The final two situations, from item B, conclude our analysis.

Proposition 8. *Let $a_1 = 0$, $c_2 = 0$, and $c_1, a_2 \in \mathbb{R} \setminus \{0\}$. Then, the following hold:*

I. i') $b > 0$, $a_2 > 0$.

If $c_1 > 0$, *then* $\mathcal{S} = [(0,0), \bar{x}_0']$;

if $c_1 < 0$, *then* $\mathcal{S} = \emptyset$;

ii') $b \geq 0$, $a_2 < 0$. $A = \mathbb{R}^2_+$.

We have $\mathcal{S} = \begin{cases} \{0\} \times [0, +\infty), & c_1 > 0 \\ \emptyset, & c_1 < 0; \end{cases}$

II. i') $b < 0$, $a_2 > 0$; $A = \emptyset$, $\mathcal{S} = \emptyset$.

ii') $b < 0$, $a_2 < 0$.

We have $\mathcal{S} = \begin{cases} \{0\} \times [b/a_2, +\infty), & c_1 > 0 \\ \emptyset, & c_1 < 0. \end{cases}$

Proposition 9. *Let $a_2 = 0$, $c_1 = 0$, and $a_1, c_2 \in \mathbb{R} \setminus \{0\}$. Then, the following hold:*

I. i') $b > 0$, $a_1 > 0$.

If $c_2 > 0$, *then* $\mathcal{S} = [(0,0), \bar{x}_0]$;

if $c_2 < 0$, *then* $\mathcal{S} = \emptyset$;

ii') $b \geq 0$, $a_1 < 0$. $A = \mathbb{R}^2_+$.

We have $\mathcal{S} = \begin{cases} [0, +\infty) \times \{0\}, & c_2 > 0 \\ \emptyset, & c_2 < 0; \end{cases}$

II. i') $b < 0$, $a_1 > 0$; $A = \emptyset$, $\mathcal{S} = \emptyset$.

ii') $b < 0$, $a_1 < 0$.

We have $\mathcal{S} = \begin{cases} [b/a_1, +\infty) \times \{0\}, & c_2 > 0 \\ \emptyset, & c_2 < 0. \end{cases}$

Remark 3. *The easiest case could be for $c_2 = 0$ and $a_2 = 0$, $a_1, c_1 \in \mathbb{R} \setminus \{0\}$. Note that $\mathcal{S} = \mathcal{S}_1 \times [0, +\infty)$, where $\mathcal{S}_1$ is the set of solutions for* $\begin{cases} c_1 \cdot x_1 \longrightarrow \min \\ a_1 \cdot x_1 \leq b \\ x_1 \geq 0. \end{cases}$ *The admissible set is*

$$A_1 = \begin{cases} [0, b/a_1], & a_1 > 0, b \geq 0 \\ [b/a_1, +\infty), & a_1 < 0, b < 0 \\ [0, +\infty), & a_1 < 0, b \geq 0 \\ \emptyset, & a_1 > 0, b < 0, \end{cases}$$

therefore $\mathcal{S}_1 =$

$$\begin{cases} \begin{cases} \{0\}, & c_1 > 0, \\ \{b/a_1\}, & c_1 < 0, \end{cases} & a_1 > 0, b \geq 0; \\ \begin{cases} \{b/a_1\}, & c_1 > 0, \\ \emptyset, & c_1 < 0, \end{cases} & a_1 < 0, b < 0 \\ \begin{cases} \{0\}, & c_1 > 0, \\ \emptyset, & c_1 < 0, \end{cases} & a_1 < 0, b \geq 0 \\ \emptyset, & a_1 > 0, b < 0. \end{cases}$$

In a similar way $\mathcal{S}$ can be obtained for $c_1 = 0$ and $a_1 = 0$, $a_2, c_2 \in \mathbb{R} \setminus \{0\}$.

3 Examples

Naturally, for each case an example is within reach. Totally, more than $3^4 \cdot 2$ of them, would cover all, therefore we unfold only a few.

Example 1. *Let us consider the problem*

$$\begin{cases} -x_1 - x_2 \longrightarrow \min \\ x_1 + a_2 \cdot x_2 \leq 1 \\ x_1, x_2 \geq 0. \end{cases}$$

Collecting data $c_1 = c_2 = -1$, $d = \begin{vmatrix} -1 & -1 \\ 1 & a_2 \end{vmatrix} = -a_2 + 1$, going through the results we get

$$\mathcal{S} = \begin{cases} \{(1,0)\}, & a_2 > 1 \\ \{(1-\alpha, \alpha) \mid \alpha \in [0,1]\}, & a_2 = 1 \\ \{(0, a_2)\}, & 0 < a_2 < 1 \\ \emptyset, & a_2 \leq 0. \end{cases}$$

Example 2. *For the parametric problems* $\begin{cases} c_1 \cdot x_1 + x_2 \longrightarrow \min \\ x_1 - x_2 \leq 1 \\ x_1, x_2 \geq 0, \end{cases}$ *we have*

$$\mathcal{S} = \begin{cases} \emptyset, & c_1 < -1, \\ \{(\alpha + 1, \alpha) \mid \alpha \geq 0\}, & c_1 = -1 \\ \{(1,0)\}, & -1 < c_1 < 0 \\ [0,1] \times \{0\}, & c_1 = 0 \\ \{(0,0)\}, & 0 < c_1. \end{cases}$$

Example 3. *Let us consider the problem*

$$\begin{cases} x_1 \longrightarrow \min \\ a_1 \cdot x_1 - x_2 \leq -1 \\ x_1, x_2 \geq 0. \end{cases}$$

For any $a_1 \in \mathbb{R}$ one obtains $\mathcal{S} = \{0\} \times [1, +\infty)$.

Example 4. *For the parametric problems* $\begin{cases} c_1 \cdot x_1 \longrightarrow \min \\ x_1 + x_2 \leq 1 \\ x_1, x_2 \geq 0, \end{cases}$ *we have* $\mathcal{S} = \begin{cases} \{0\} \times [0, +\infty), & c_1 > 0 \\ co\{(0,0), (1,0), (0,1)\}, & c_1 = 0 \\ \{(1,0)\}, & c_1 < 0, \end{cases}$ *for* $\begin{cases} c_1 \cdot x_1 \longrightarrow \min \\ x_1 - x_2 \leq 1 \\ x_1, x_2 \geq 0, \end{cases}$ *one has*

$$\mathcal{S} = \begin{cases} \{0\} \times [0, +\infty), & c_1 > 0 \\ \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 - x_2 \leq 1\}, & c_1 = 0 \\ \emptyset, & c_1 < 0, \end{cases}$$

and $\begin{cases} c_1 \cdot x_1 \longrightarrow \min \\ -x_1 + x_2 \le 1 \\ x_1, x_2 \ge 0, \end{cases}$ *has*

$$\mathcal{S} = \begin{cases} \{0\} \times [0,1], & c_1 > 0 \\ \{(x_1,x_2) \in \mathbb{R}^2_+ \mid -x_1 + x_2 \le 1\}, & c_1 = 0 \\ \emptyset, & c_1 < 0. \end{cases}$$

All the cases structured in A, $A1$, $A11$, $A12$, $A2$, $A3$, $A31$, $A32$, and B are contained in a source code (see a party of it in figure 3 below) and a sample of the running software (see figure 4):

```
#include <iostream>
#include <math.h>

using namespace std;
int main()
{
    int a[2], c[2], d;
    int n, i;
    float b;

    cout<<"read the components of vector c:"<<endl;
    for(i=0;i<2;i++)
    {   cout<<"c["<<i<<"] = ";
        cin>>c[i];
        cout<<endl;
    }
    cout<<"vector c has components:"<<endl;
    for(i=0;i<2;i++)
    {
        cout<<"c["<<i<<"] = "<<c[i]<<endl;
    }
    cout<<"read the components of vector a:"<<endl;
    for(i=0;i<2;i++)
    {   cout<<"a["<<i<<"] = ";
        cin>>a[i];
        cout<<endl;
    }
    cout<<"vector c has components:"<<endl;
    for(i=0;i<2;i++)
    {
        cout<<"a["<<i<<"] = "<<a[i]<<endl;
    }
    d=c[0]*a[1]-c[1]*a[0];
    cout<<"the system determinant is: d = "<<d<<endl;
    cout<<"read the free term b:";
    cin>>b;
    cout<<endl;

    if(c[0]*c[1]!=0){
    if(d!=0){
    if(b>=0){
    if (a[0]>0&&a[1]>0)
    {   cout<<"This case corresponds to I. i)\n";
```

Figure 3. source code

To comment on CASystems, the Python script uses PuLP (https://pythonhosted.org/PuLP/). Before using optimization with PuLP, the user needs to get access to *pulp* library via Anaconda Prompt *pip install pulp*. Mainly, *solve* function is called for finding a solution of the linear programming problem. After the script is running, the output is a/the vector solution of the problem. For the linear programming problem $\begin{cases} -x_1 - x_2 \longrightarrow \min \\ x_1 + x_2 \le 1 \\ x_1, x_2 \ge 0, \end{cases}$ with

$$\mathcal{S} = \{(1-\alpha, \alpha) \mid \alpha \in [0,1]\},$$

```
read the components of vector c:
c[0] = 1

c[1] = 2

vector c has components:
c[0] = 1
c[1] = 2
read the components of vector a:
a[0] = 2

a[1] = 4

vector c has components:
a[0] = 2
a[1] = 4
the system determinant is: d = 0
read the free term b:5

This case corresponds to I. i)
S = {(0, 0)}
```

Figure 4. running software

the commands required are:

import pulp $*$
$prob$
$= pulp.LpProblem(pulp.LpMaximize)$
$x1 = pulp.LpVariable('x1', lowBound = 0)$
$x2 = pulp.LpVariable('x2', lowBound = 0)$
$prob += x1 + x2$
$prob += x1 + x2 <= 1$
$prob.solve()$

By calling the solution values of the components with the commands:

$$x1.varValue$$

1.0

$$x2.varValue$$

and 0.0 are returned, respectively.

In a similar way, for the linear programming problem $\begin{cases} -x_1 + x_2 \longrightarrow \min \\ x_1 - x_2 \le 0 \\ x_1, x_2 \ge 0, \end{cases}$ with

$$\mathcal{S} = \{(\alpha, \alpha) \mid \alpha \in [0, +\infty)\},$$

the returned values of the solution components are 0.0 and 0.0, respectively.

4 Conclusions and future work

For given data a, b, c our programm returns the set of solutions (see for example, figure 4). It would be interesting to be able to select the answer when one of the inputs is given as parameter.

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