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## EXISTENCE OF POSITIVE SOLUTION FOR A SINGULAR FOURTH-ORDER DIFFERENTIAL SYSTEM

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### Abstract

*This paper investigates the existence of positive solutions for a fourth-order differential system using a fixed point theorem of cone expansion and compression type.*

### 1. Introduction

It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \quad (1)$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \quad (2)$$

Existence of solutions for problem (1) was established for example by Gupta [1,2], Liu [3], Ma [4], Ma et. al. [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9], RP Agarwal et.al. [10,11,12] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [13] studied the existence of positive solutions for the second-order boundary value problem

$$\begin{aligned} -u'' + \lambda u &= u\varphi + f(t, u), \quad 0 < t < 1 \\ -\varphi'' &= \mu u, \quad 0 < t < 1 \\ u(0) &= u(1) = 0, \\ \varphi(0) &= \varphi(1) = 0, \end{aligned} \quad (3)$$

where  $\lambda > -\pi^2$ ,  $\mu$  is a positive parameter, and  $f(t, u) : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is continuous.

In this paper we shall discuss the existence of positive solutions for the fourth-order boundary value problem

$$u^{(4)} = \varphi u + f(t, u, u''), \quad 0 < t < 1$$

$$-\varphi'' = \mu g(t, u, u''), \quad 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

$$\varphi(0) = \varphi(1) = 0, \quad (4)$$

where  $\mu$  is a positive parameter and  $f(t, u, v) : (0, 1) \times [0, \infty) \times (-\infty, 0] \rightarrow [0, \infty)$  is continuous. In fact as we will see below one could consider in Section 2 and 3  $f(t, u, v) \leq f_1(t) f_2(t, u, v)$  with  $f_2(t, u, v) : [0, 1] \times [0, \infty) \times (-\infty, 0] \rightarrow [0, \infty)$  continuous and  $f_1 \in C[(0, 1), \mathbb{R}^+]$  provided

$$\int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) ds d\tau < +\infty;$$

here  $K$  is as defined in Section 2. Moreover, our hypotheses allow but do not require  $g(t, u, v) : [0, 1] \times [0, \infty) \times (-\infty, 0] \rightarrow [0, \infty)$  to be singular at  $v = 0$ .

## 2. Preliminaries

Let  $Y = C[0, 1]$  and  $Y_+ = \{u \in Y : u(t) \geq 0, t \in [0, 1]\}$ . It is well known that  $Y$  is a Banach space equipped with the norm  $\|u\|_0 = \sup_{t \in [0, 1]} |u(t)|$ . We denote the norm  $\|u\|_2$  by

$$\|u\|_2 = \max \{ \|u\|_0, \|u''\|_0 \}.$$

It is easy to show that  $C^2[0, 1]$  is complete with the norm  $\|u\|_2$  and  $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2 \|u\|_2$ .

Suppose that  $K(t, s)$  is the Green function associated with

$$-u'' = f(t), \quad u(0) = u(1) = 0, \quad (5)$$

which is explicitly expressed by

$$K(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

We need the following lemmas.

**Lemma 1.**  $K(t, s)$  has the following properties:

- (i)  $K(t, s) > 0, \forall t, s \in (0, 1)$ ;
- (ii)  $K(t, s) \leq K(s, s), \forall t, s \in [0, 1]$ ;
- (iii)  $K(t, s) \geq K(t, t)K(s, s), \forall t, s \in [0, 1]$ ;
- (iv)  $|K(t_1, s) - K(t_2, s)| \leq |t_1 - t_2|$ , for all  $t_1, t_2, s \in [0, 1]$

**Lemma 2** ([14]). Let  $E$  be a real Banach space and let  $P \subset E$  be a cone in  $E$ . Assume  $\Omega_1, \Omega_2$  are open subset of  $E$  with  $\theta \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ , and let  $Q : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$  be a completely continuous operator such that either

- (i)  $\|Qu\| \leq \|u\|, u \in P \cap \partial\Omega_1$  and  $\|Qu\| \geq \|u\|, u \in P \cap \partial\Omega_2$ ; or
- (ii)  $\|Qu\| \geq \|u\|, u \in P \cap \partial\Omega_1$  and  $\|Qu\| \leq \|u\|, u \in P \cap \partial\Omega_2$ .

Then  $Q$  has a fixed point in  $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

The boundary value problem

$$-\varphi'' = \mu g(t, u(t), u''(t)), \quad \varphi(0) = \varphi(1) = 0,$$

can be solved by using the Green's function, namely,

$$\varphi(t) = \mu \int_0^1 K(t, s) g(s, u(s), u''(s)) ds, \quad 0 < t < 1. \quad (6)$$

Thus inserting (6) into the first equation of (4), we have

$$u^{(4)} = \mu u(t) \int_0^1 K(t, s) g(s, u(s), u''(s)) ds + f(t, u, u''),$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \quad (7)$$

Now we consider the existence of a positive solution of (7). The function  $u \in C^4(0, 1) \cap C^2[0, 1]$  is a positive solution of (7), if  $u \geq 0$ ,  $t \in [0, 1]$ , and  $u \neq 0$ .

Then the solution of (7) can be expressed as

$$u(t) = \mu \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds d\tau +$$

$$+ \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) f(s, u(s), u''(s)) ds d\tau \quad (8)$$

and the second-order derivative  $u''$  can be expressed by

$$u''(t) = -\mu \int_0^1 K(t, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds -$$

$$- \int_0^1 K(t, s) f(s, u(s), u''(s)) ds. \quad (9)$$

Set

$$P = \{u \in C^2[0, 1] : u(0) = u(1) = 0, u(t) \geq K(t, t) \|u\|_0, -u''(t) \geq K(t, t) \|u''\|_0, t \in [0, 1]\}.$$

Note  $P$  is a cone in  $C^2[0, 1]$ . For  $R > 0$ , write  $B_R = \{u \in C^2[0, 1] : \|u\|_2 < R\}$ .

We now define a mapping  $T : P \rightarrow C^2[0, 1]$  by

$$Tu(t) = \mu \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds d\tau +$$

$$+ \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) f(s, u(s), u''(s)) ds d\tau. \quad (10)$$

It is easy to see that if  $u \in P$  then

$$-u''(t) \geq \sigma \|u''\|_0, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad (11)$$

where  $\sigma = \frac{3}{16}$ .

**Lemma 3.** Let  $w \in P$ . Then the following relations hold:

- (a)  $(Tw)(t) \geq K(t, t) \|Tw\|_0$  for  $t \in [0, 1]$ , and
- (b)  $-(Tw)''(t) \geq K(t, t) \|Tw''\|_0$  for  $t \in [0, 1]$ .

**Proof.** For simplicity we denote

$$\begin{aligned}
I &= \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) w(s) \int_0^1 K(s, v) q(v) dv ds d\tau + \\
&\quad + \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) h(s) ds d\tau, \\
J &= \mu \int_0^1 K(s, s) w(s) \int_0^1 K(s, v) q(v) dv ds + \\
&\quad + \int_0^1 K(s, s) h(s) ds,
\end{aligned}$$

and

$$q(v) = g(v, w(v), w''(v)), \quad h(s) = f(s, w(s), w''(s)).$$

From Lemma 1 it is easy to see that

$$K(t, t)I \leq Tw(t) \leq I \quad \text{and} \quad K(t, t)J \leq -(Tw)''(t) \leq J, \quad t \in [0, 1] \quad (12)$$

Using (12), we have

$$\|Tw\|_0 \leq I, \quad \text{and} \quad \|-(Tw)''\|_0 \leq J,$$

hence

$$(Tw)(t) \geq K(t, t) \|Tw\|_0 \quad \text{and} \quad -(Tw)''(t) \geq K(t, t) \|Tw''\|_0 \quad \text{for } t \in [0, 1].$$

Throughout this paper, we assume additionally that the function  $f(t, u, v)$  satisfies (H1)

$$f(t, u, v) \leq f_1(t) f_2(u + |v|), \quad t \in (0, 1), \quad u \in R^+, \quad v \in R^-,$$

where  $f_1(t) \in C[(0, 1), R^+]$ ,  $f_2(w) \in C[R^+, R^+]$ .

Moreover the function  $g(t, u, v) : [0, 1] \times (0, +\infty) \times (-\infty, 0) \rightarrow [0, +\infty)$  satisfies

(H2) There exists an  $a > 0$  such that  $g(t, u, v)$  is nondecreasing in  $v \leq a$  for each fixed  $t \in [0, 1]$  and  $u \in R^+$  i.e. if  $-a \leq v_2 \leq v_1 < 0$  then  $g(t, u, v_1) \geq g(t, u, v_2)$ .

(H3) For each fixed  $0 < r \leq a$

$$0 < \int_0^1 g(s, u, rs(1-s)) ds < \infty, \quad \forall u \in [0, +\infty).$$

Let us introduce the following notations

$$D_1 = \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) ds d\tau, \quad D_4 = \int_0^1 K(s, s) f_1(s) ds,$$

$$D_3 = \int_0^1 K(\tau, \tau) d\tau, \quad D_2 = \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) ds d\tau,$$

$$D_5 = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) ds d\tau.$$

**Lemma 4.** Let (H1), (H2), and (H3) hold. Then for all  $u \in P \cap \overline{B}_R/B_r$  where  $r < a < R$  the following hold

$$(Tu)(t) \leq \mu D_3 \|u\|_0 M + D_4 \sup_{s \in (0,1)} f_2(u(s) + |u''(s)|), \quad t \in (0,1),$$

and

$$-(Tu)''(t) \leq \mu D_3 \|u\|_0 M + D_4 \sup_{s \in (0,1)} f_2(u(s) + |u''(s)|), \quad t \in (0,1),$$

where

$$\begin{aligned} M = & \sup_{w \in [0,R]} \int_0^1 K(v,v) g(v,w, -rK(v,v)) dv + \\ & + \sup_{w \in [0,R]} \sup_{z \in [a,R]} \int_0^1 K(v,v) g(v,w,z) dv. \end{aligned}$$

**Proof.** It is easy to see that  $D_1 \leq D_3$ , and  $D_2 \leq D_4$ . Let  $u \in P \cap \overline{B}_R/B_r$ , then by Lemma 6,  $\|u\|_0 \leq \|u''\|_0$  and by Corollary 7,  $\|u\|_2 = \|u''\|_0$ . Thus  $r \leq \|u''\|_0 \leq R$ . Also since  $u \in P$  we have  $-u''(t) \geq K(t,t) \|u''\|$ ,  $t \in [0,1]$ .

By Lemma 1. and (H1) – (H3) we have

$$\begin{aligned} Tu(t) &= \mu \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) u(s) \int_0^1 K(s,v) g(v,u(v), u''(v)) dv ds d\tau + \\ &+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) f(s,u(s), u''(s)) ds d\tau = \\ &= \mu \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) u(s) \int_{|u''(v)| \leq a} K(s,v) g(v,u(v), u''(v)) dv ds d\tau + \\ &+ \mu \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) u(s) \int_{|u''(v)| \geq a} K(s,v) g(v,u(v), u''(v)) dv ds d\tau + \\ &+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) f(s,u(s), u''(s)) ds d\tau \leq \\ &\leq \mu \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) \|u\|_0 \sup_{w \in [0,R]} \int_{|u''(v)| \leq a} K(v,v) g(v,w, -rK(v,v)) dv ds d\tau + \\ &+ \mu \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) \|u\|_0 \left[ \sup_{w \in [0,R]} \sup_{z \in [a,R]} \int_{|u''(v)| \geq a} K(v,v) g(v,w,z) dv \right] ds d\tau + \\ &+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s) ds d\tau \sup_{s \in (0,1)} f_2(u(s) + |u''(s)|) \end{aligned}$$

$$\begin{aligned}
&\leq \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) \|u\|_0 \sup_{w \in [0, R]} \int_0^1 K(v, v) g(v, w, -rK(v, v)) dv ds d\tau + \\
&+ \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) \|u\|_0 \left[ \sup_{w \in [0, R]} \sup_{z \in [a, R]} \int_0^1 K(v, v) g(v, w, z) dv \right] ds d\tau + \\
&\quad + \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) ds d\tau \sup_{s \in (0, 1)} f_2(u(s) + |u''(s)|) \\
&\quad \leq \mu D_1 M \|u\|_0 + D_2 \sup_{s \in (0, 1)} f_2(u(s) + |u''(s)|), \\
&\quad \leq \mu D_3 M \|u\|_0 + D_4 \sup_{s \in (0, 1)} f_2(u(s) + |u''(s)|),
\end{aligned}$$

and similarly we also have

$$\begin{aligned}
-(Tu)''(t) &\leq \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) ds d\tau \|u\|_0 M + \\
&+ \int_0^1 K(s, s) f_1(s) ds d\tau \sup_{s \in (0, 1)} f_2(u(s) + |u''(s)|) \\
&\leq \mu D_3 \|u\|_0 M + D_4 \sup_{s \in (0, 1)} f_2(u(s) + |u''(s)|).
\end{aligned}$$

**Lemma 5.**  $T(P) \subset P$  and  $T : P \rightarrow P$  is completely continuous.

**Proof.** Let  $u \in P$ , then we define mapping  $T : P \rightarrow C^2[0, 1]$  by (10). Then for any  $u \in P$ , it is clear that

$$\begin{aligned}
(Tu)''(t) &= -\mu \int_0^1 K(t, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds - \\
&\quad - \int_0^1 K(t, s) f(s, u(s), u''(s)) ds \leq 0.
\end{aligned} \tag{13}$$

By Lemma 3,

$$Tu(t) \geq K(t, t) \|Tu\|_0 \geq \sigma \|Tu\|_0, \quad t \in [0, 1]$$

and

$$-(Tu)''(t) \geq K(t, t) \|(Tu)''\|_0 \geq \sigma \|(Tu)''\|_0, \quad t \in [0, 1].$$

Hence  $T(P) \subset P$ .

Let  $V \subset P$  be a bounded set. Then there exists a  $d > 0$ , such that  $\sup\{\|u\|_2 : u \in V\} = d$ .

First we prove  $T(V)$  is bounded. Since  $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ , we have  $u(t) + |u''(t)| \leq \|u\|_0 + \|u''\|_0 \leq 2d$ , for all  $t \in [0, 1]$ . Let  $M_d = \sup\{f_2(w) : w \in [0, 2d]\}$ . Now from Lemma 4 we have for any  $u \in V$  and  $t \in [0, 1]$  that

$$\begin{aligned}
|Tu(t)| &= \left| \mu \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds d\tau \right. \\
&\quad \left. + \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) f(s, u(s), u''(s)) ds d\tau \right| \\
&\leq \mu D_1 \|u\|_0 M + D_2 \sup_{s \in (0,1)} f_2(u(s) + |u''(s)|) \\
&\leq \mu D_1 dM + M_d D_2. \tag{14}
\end{aligned}$$

We have a similar type inequality for  $|(Tu)''(t)|$ .

Therefore  $T(V)$  is bounded.

Next we prove that  $T(V)$  is equicontinuous. Now from Lemma 4 we have for any  $u \in V$  and any  $t_1, t_2 \in [0, 1]$  that

$$\begin{aligned}
& |(Tu)(t_1) - (Tu)(t_2)| \leq \\
& \leq \mu \int_0^1 \int_0^1 |K(t_1, \tau) - K(t_2, \tau)| K(\tau, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds d\tau \\
& \quad + \int_0^1 \int_0^1 |K(t_1, \tau) - K(t_2, \tau)| K(\tau, s) f(s, u(s), u''(s)) ds d\tau \\
& \leq \mu \int_0^1 \int_0^1 |K(t_1, \tau) - K(t_2, \tau)| K(\tau, s) ds d\tau \|u\|_0 M \\
& \quad + \int_0^1 \int_0^1 |K(t_1, \tau) - K(t_2, \tau)| K(\tau, s) f_1(s) f_2(u(s) + |u''(s)|) ds d\tau \\
& \leq \mu |t_1 - t_2| \int_0^1 \int_0^1 K(s, s) K(s, v) dv ds \|u\|_0 M \\
& \quad + M_d |t_1 - t_2| \int_0^1 \int_0^1 K(s, s) f_1(s) ds d\tau \leq (\mu D_3 dM + M_d D_4) |t_1 - t_2|.
\end{aligned}$$

We have a similar type inequality for  $|(Tu)''(t_1) - (Tu)''(t_2)|$ .

Therefore  $T(V)$  is equicontinuous.

Next we prove that  $T$  is continuous. Suppose  $u_n, u \in P$  and  $\|u_n - u\|_2 \rightarrow 0$  which implies that  $u_n(t) \rightarrow u(t), u_n''(t) \rightarrow u''(t)$  uniformly on  $[0, 1]$ . Similarly for  $f(t, u, v) \leq f_1(t) f_2(|u| + |v|), f_2(|u_n(t)| + |u_n''(t)|) \rightarrow f_2(|u(t)| + |u''(t)|)$  uniformly on  $[0, 1]$  and  $g(t, u_n(t)) \rightarrow g(t, u(t))$  uniformly on  $[0, 1]$ . The assertion follows from the estimate

$$\begin{aligned}
& |Tu_n(t) - Tu(t)| \leq \\
& \leq \mu \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) \left| u_n(s) \int_0^1 K(s, v) g(v, u_n(v), u_n''(v)) dv - u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv \right| ds d\tau
\end{aligned}$$

$$+ \int_0^1 \int_0^1 K(t, \tau) K(\tau, s) |f_1(s)| |f_2(|u_n(s)| + |u_n''(s)|) - f_2(|u(s)| + |u''(s)|)| ds d\tau$$

and the similar estimate for  $|(Tu_n)''(t) - (Tu)''(t)|$  by an application of the standard theorem on the convergence of integrals.

The Ascoli-Arzelà theorem guarantees that  $T : P \rightarrow P$  is completely continuous.

**Lemma 6.** If  $u(0) = u(1) = 0$  and  $u \in C^2[0, 1]$ , then  $\|u\|_0 \leq \|u''\|_0$ , and so,  $\|u\|_2 = \|u''\|_0$ .

**Proof.** Since  $u(0) = u(1) = 0$ , there is a  $\alpha \in (0, 1)$  such that  $u'(\alpha) = 0$ , and so  $u'(t) = \int_\alpha^t u''(s) ds$ ,  $t \in [0, 1]$ . Hence  $|u'(t)| \leq \int_\alpha^t |u''(s)| ds \leq \int_0^1 |u''(s)| ds \leq \|u''\|_0$ ,  $t \in [0, 1]$ . Thus  $\|u'\|_0 \leq \|u''\|_0$ . Since  $u(0) = 0$ , we have  $u(t) = \int_0^t u'(s) ds$ ,  $t \in [0, 1]$ , and so  $|u(t)| \leq \int_0^1 |u'(s)| ds \leq \|u'\|_0$ . Thus  $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$ . Since  $\|u\|_2 = \max\{\|u\|_0, \|u'\|_0\}$  and  $\|u\|_0 \leq \|u''\|_0$ , we obtain that  $\|u\|_2 = \|u''\|_0$ .

**Corollary 7.** Let  $r > 0$  and let  $u \in \partial B_r \cap P$ . Then  $\|u\|_2 = \|u''\|_0 = r$ .

### 3. Main results

**Theorem 1.** Let (H1),(H2) and (H3) hold. Assume that the following condition holds (H4)

$$\limsup_{w \rightarrow 0^+} \frac{f_2(w)}{w} = 0,$$

and

$$\liminf_{|v| \rightarrow \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \inf_{u \in [0, +\infty)} \frac{f(t, u, v)}{|v|} = \infty.$$

If  $\mu \in (0, \frac{1}{4D_1M})$ , then problem (4) has at least one positive solution.

**Proof.** Let us choose  $0 < c_1 \leq \frac{1}{8D_2}$ . Then by (H4), there exist  $0 < r < \frac{a}{2}$  such that

$$f_2(u + |v|) \leq c_1 (u + |v|), \quad 0 \leq u + |v| \leq 2r.$$

Let  $u \in \partial B_r \cap P$ , then by Corollary 7,  $\|u\|_2 = \|u''\|_0 = r$  and  $u(0) = u(1) = 0$ . Also since  $\|u\|_0 \leq \|u''\|_0$  we have  $u(t) \leq \|u\|_0 \leq r$ ,  $|u''(t)| \leq \|u''\|_0 = r$ ,  $\forall t \in [0, 1]$ . Thus  $0 \leq u(t) + |u''(t)| \leq 2r$ ,  $\forall t \in [0, 1]$ .

Thus, by Lemma 4, and (H1 – H2) we have

$$\begin{aligned} (Tu)(t) &\leq \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) u(s) ds d\tau \int_0^1 K(v, v) g(v, u(v), -rK(v, v)) dv + \\ &+ \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) f_2(u(s) + |u''(s)|) ds d\tau \\ &\leq \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) ds d\tau \|u\|_0 M + \\ &+ \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) c_1 (u(s) + |u''(s)|) ds d\tau \end{aligned}$$



$$\begin{aligned}
&\leq \mu D_1 \|u\|_0 M + c_1 D_2 (\|u\|_0 + \|u''\|_0) \\
&\leq \frac{1}{4} \|u\|_0 + \frac{1}{4} \|u\|_2 \leq \frac{1}{4} \|u\|_2 + \frac{1}{4} \|u\|_2 \leq \\
&\leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_r \cap P, \quad t \in [0, 1],
\end{aligned}$$

where

$$M = \sup_{w \in [0, a]} \int_0^1 K(v, v) g(v, w, -rK(v, v)) dv.$$

Consequently,

$$\|Tu\|_0 \leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_r \cap P. \quad (15)$$

Similarly we also have

$$\begin{aligned}
(Tu)''(t) &= -\mu \int_0^1 K(t, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds - \\
&\quad - \int_0^1 K(t, s) f(t, u(s), u''(s)) ds.
\end{aligned}$$

Hence

$$\begin{aligned}
|(Tu)''(t)| &\leq \mu \int_0^1 K(s, s) ds \|u\|_0 M + \\
&\quad + \int_0^1 K(s, s) f_1(s) f_2(|u| + |u''|) ds \\
&\leq \mu D_3 \|u\|_0 M + c_1 D_4 (\|u\|_0 + \|u''\|_0) \\
&\leq \mu D_1 \|u\|_0 M + c_1 D_2 (\|u\|_0 + \|u''\|_0) \\
&\leq \frac{1}{4} \|u\|_0 + \frac{1}{4} \|u\|_2 \leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_r \cap P, \quad t \in [0, 1].
\end{aligned}$$

Consequently,

$$\|(Tu)''\|_0 \leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_r \cap P. \quad (16)$$

Using (15) and (16) we have

$$\|Tu\|_2 \leq \|Tu\|_0 + \|(Tu)''\|_0 \leq \|u\|_2, \quad \forall u \in \partial B_r \cap P. \quad (17)$$

Let us choose  $c_2 \geq \frac{1}{\sigma D_3}$ . Then by condition (H4), there exists  $R_1 > 0$  such that

$$f(t, u, v) \geq c_2 |v|, \quad \forall u \in R^+, \quad \forall |v| \geq R_1, \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right].$$

Let  $R > \max\{\frac{R_1}{\sigma}, a\}$ . Let  $u \in \partial B_R \cap P$ , i.e.  $\|u''\|_0 = R$ . Thus by using (11) we have

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} |u''(t)| \geq \sigma \|u''\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 1, we have

$$\begin{aligned} (Tu)\left(\frac{1}{2}\right) &\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) u(s) K(s, v) g(v, u(v), u''(v)) dv ds d\tau \\ &\quad + \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) f(t, u(s), u''(s)) ds d\tau \\ &\geq c_2 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) |u''(s)| ds d\tau \\ &\geq c_2 \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) ds d\tau \|u''\|_0 \geq \|u''\|_0 \end{aligned}$$

so

$$(Tu)\left(\frac{1}{2}\right) \geq \|u''\|_0 = \|u\|_2, \quad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \quad \forall u \in \partial B_R \cap P.$$

Then due to Lemma 2, by (17) and the above inequality we see that the problem (4) has at least one positive solution.

**Theorem 2.** Let (H1),(H2) and (H3) hold. Assume that the following conditions hold

(H5)

$$\liminf_{|u|+|v| \rightarrow 0^+} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, u, v)}{|u| + |v|} = \infty,$$

and

$$\liminf_{|v| \rightarrow \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \inf_{u \in [0, +\infty)} \frac{f(t, u, v)}{|v|} = \infty.$$

(H6) there exists  $0 < \varrho < \frac{a}{2}$  such that

$$\sup_{w \in [0, a]} f_2(w) \leq \frac{\varrho}{4D_4}. \quad (18)$$

If  $\mu \in (0, \frac{1}{4D_3M})$ , then problem (4) has at least two positive solutions.

We note for the argument below that  $D_1 \leq D_3$  and  $D_2 \leq D_4$ .

**Proof.** By condition (H6) there exists  $0 < \varrho < \frac{a}{2}$  such that (18) is fulfilled. Let  $u \in \partial B_\varrho \cap P$ , by Corollary 7,  $\|u''\|_0 = \varrho$ ,  $u(0) = u(1) = 0$ . Also since  $\|u\|_0 \leq \|u''\|_0$  we have  $u(t) \leq \|u\|_0 \leq \varrho$ ,  $|u''(t)| \leq \|u''\|_0 = \varrho$ ,  $\forall t \in [0, 1]$ . Thus  $0 \leq |u(t)| + |u''(t)| < 2\varrho$ ,  $\forall t \in [0, 1]$ . By condition (H6),  $\forall u \in \partial B_\varrho \cap P$  and  $t \in [0, 1]$ , we have

$$\begin{aligned}
(Tu)(t) &\leq \mu \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) ds d\tau \|u\|_0 M + \\
&+ \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) f_2(u(s) + |u''(s)|) ds d\tau \\
&\leq \mu D_1 \|u\|_0 M + \frac{\varrho}{4D_4} \int_0^1 \int_0^1 K(\tau, \tau) K(\tau, s) f_1(s) ds \\
&\leq \frac{1}{4} \|u\|_0 + \frac{1}{4} \varrho = \frac{1}{4} \|u\|_0 + \frac{1}{4} \|u''\|_0 = \frac{1}{4} \|u\|_0 + \frac{1}{4} \|u\|_2 \leq \\
&\leq \frac{1}{4} \|u\|_2 + \frac{1}{4} \|u\|_2 \leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_\varrho \cap P, \quad t \in [0, 1],
\end{aligned}$$

where

$$M = \sup_{w \in [0, a]} \int_0^1 K(v, v) g(v, w, -\varrho v(1-v)) dv.$$

Consequently, we get

$$\|Tu\|_0 \leq \|u\|_2, \quad \forall u \in \partial B_\varrho \cap P. \quad (19)$$

Similarly we also have

$$\begin{aligned}
(Tu)''(t) &= -\mu \int_0^1 K(t, s) u(s) \int_0^1 K(s, v) g(v, u(v), u''(v)) dv ds d\tau - \\
&- \int_0^1 K(t, s) f(s, u(s), u''(s)) ds d\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
|(Tu)''(t)| &\leq \mu \int_0^1 K(s, s) ds \|u\|_0 M + \int_0^1 K(s, s) f_1(s) f_2(u(s) + |u''(s)|) ds \\
&\leq \mu D_3 \|u\|_0 M + \frac{\varrho}{4D_4} \int_0^1 K(s, s) f_1(s) ds \\
&\leq \frac{1}{4} \|u\|_0 + \frac{1}{4} \varrho = \frac{1}{4} \|u\|_0 + \frac{1}{4} \|u\|_2 \leq \\
&\leq \frac{1}{4} \|u\|_2 + \frac{1}{4} \|u\|_2 \leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_\varrho \cap P, \quad t \in [0, 1].
\end{aligned}$$

Consequently,

$$\|(Tu)''\|_0 \leq \frac{1}{2} \|u\|_2, \quad \forall u \in \partial B_\rho \cap P. \quad (20)$$

Using (19) and (20) we have

$$\|Tu\|_2 \leq \|Tu\|_0 + \|(Tu)''\|_0 \leq \|u\|_2, \quad \forall u \in \partial B_\rho \cap P. \quad (21)$$

Let us choose  $0 < c_3 \leq \frac{1}{\sigma D_5}$ . The by condition (H5), there exists  $0 < r < \varrho$  such that

$$f(t, u, v) \geq c_3 (u + |v|), \quad \forall u \in [0, r], \quad \forall |v| \in [0, r], \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let  $u \in \partial B_r \cap P$ , by Corollary 7,  $\|u''\|_0 = r$ ,  $u(0) = u(1) = 0$ . Also since  $\|u\|_0 \leq \|u''\|_0$  we have

$$0 \leq u(t) \leq \|u\|_0 \leq r, \quad 0 \leq |u''(t)| \leq \|u''\|_0 = \|u\|_2 = r, \quad \forall u \in \partial B_r \cap P.$$

Thus by using (11) we have

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} |u''(t)| \geq \sigma \|u''\|_0 = \sigma r, \quad \forall u \in \partial B_r \cap P.$$

The estimate for  $(Tu)(\frac{1}{2})$  is similar to that in the proof of Theorem 1 i.e. from Lemma 1 and (H5) we have

$$\begin{aligned} (Tu)\left(\frac{1}{2}\right) &\geq c_3 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) (u(s) + |u''(s)|) ds d\tau \\ &\geq c_3 \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) ds d\tau \|u''\|_0 \geq \|u''\|_0. \end{aligned}$$

Thus

$$(Tu)\left(\frac{1}{2}\right) \geq \|u''\|_0 = \|u\|_2, \quad \forall u \in \partial B_r \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \quad \forall u \in \partial B_r \cap P.$$

Finally we show that for sufficiently large  $R > a$ , it holds

$$\|Tu\|_2 \geq \|u\|_2, \quad \forall u \in \partial B_R \cap P.$$

To see this we choose  $c_2 \geq \frac{1}{\sigma D_5}$ . Due to condition (H5), there exist  $R_1 > 0$  such that

$$f(t, u, v) \geq c_2 |v|, \quad \forall u \in R^+, \quad \forall |v| \geq R_1, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let  $R > \max\{\frac{R_1}{\sigma}, a\}$ . Let  $u \in \partial B_R \cap P$ , by Corollary 7,  $\|u''\|_0 = R$ . Thus by using (11) we have

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} |u''| \geq \sigma \|u''\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 1, (H1) and (H4), we have

$$\begin{aligned} (Tu)\left(\frac{1}{2}\right) &\geq c_2 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) |u''(s)| ds d\tau \\ &\geq c_2 \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2}, \tau\right) K(\tau, s) ds d\tau \|u''\|_0 \geq \|u''\|_0 \end{aligned}$$

so

$$(Tu)\left(\frac{1}{2}\right) \geq \|u''\|_0 = \|u\|_2, \quad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \quad \forall u \in \partial B_R \cap P.$$

Then by Lemma 2, we know that  $T$  has at least two fixed points in  $(\overline{B_R} \setminus B_\rho) \cap P$  and  $(\overline{B_\rho} \setminus B_r) \cap P$ , i.e. problem (4) has at least two positive solutions.

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