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EXTENSIONS OF STEFFENSEN'S METHOD TO THE SPACE $\mathbb{R}^n$

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Abstract

The purpose of this paper is to give some extensions of Steffensen's method to the space $\mathbb{R}^n$.

Keywords: nonlinear equations, nonlinear system of equations, Banach fixed point theorem, mean value theorem for real vector-valued functions

1 Introduction

It is known the following result for iterative functions on the real line, see for example [1] or [2]

Theorem 1. *(general theorem for real iterative functions) If $\phi : J \to \mathbb{R}$ is derivable on the interval $J = [x_0 - \delta, x_0 + \delta]$, $\delta > 0$ and the derivative function ϕ' satisfies the inequality $0 \le |\phi'(x)| \le m < 1$ for every $x \in J$ and the point $x_1 = \phi(x_0)$ verifies the inequality $|x_1 - x_0| \le (1 - m)\delta$, then:*

- *we can form the sequence $\{x_k\}_{k\in\mathbb{N}}$ with the iterative rule $x_{k+1} = \phi(x_k)$, $k \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ we have $x_k \in J$;*
- *there exists $\lim_{k\to\infty} x_k = x^* \in J$;*
- *x^* is the unique solution of the equation $\phi(x) = x$ in the interval J.*

In [3] we showed a complex variant of this theorem using [4] and [5]:

Theorem 2. *(general theorem for complex iterative functions) If $\phi : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on the closed disc $B(z_0, r) \subset \mathbb{C}$, $z_0 \in \mathbb{C}$, $r > 0$, such that the derivative function ϕ' satisfies the inequality $0 \le |\phi'(z)| \le m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0, r)$ and the point $z_1 = \phi(z_0)$ verifies the inequality $|z_1 - z_0| \le (1 - \sqrt{2} \cdot m) \cdot r$, then:*

- *we can form the sequence $\{z_k\}_{k\in\mathbb{N}}$ with the iterative rule $z_{k+1} = \phi(z_k)$, $k \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ we have $z_k \in B(z_0, r)$;*
- *there exists the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ and $\lim_{k\to\infty} z_k = z^* \in B(z_0, r)$;*
- *z^* is the unique solution of the equation $\phi(z) = z$ in the closed disc $B(z_0, r)$.*

Using theorem 2 in [3] we gave some applications for nonlinear complex equations. In order to solve the complex equation $f(z) = 0$ we used the following transformations: translation, translation and homothety, the Newton's method in complex case and we builded the complex version of parallel, chord and Steffensen's method.

In [6] we showed extensions of the above mentioned methods to the space $\mathbb{R}^n$ in order to solve nonlinear system of equations. At the same time we gave a theoretical result for the convergence of these methods.

2 Main part

Let us consider the euclidian norm $||\cdot||$ on the space $\mathbb{R}^n$ and the closed sphere $B(w, r) = \{x \in$

$\mathbb{R}^n / ||x - w|| \leq r\}$ in the space $\mathbb{R}^n$, with center w and radius $r > 0$. First we remember the Banach fixed point theorem in the case of the closed sphere $B(w, r)$:

Theorem 3. *Let $\phi : B(w,r) \to B(w,r)$ be a contraction, i.e. there exists the constant $\alpha \in [0, 1)$ such that $||\phi(x) - \phi(y)|| \leq \alpha \cdot ||x - y||$ for every $x, y \in B(w, r)$. Then the function ϕ has a unique fixed point in $B(w, r)$, which can be obtained as the limit of the sequence $\{x^k\}_{k\in\mathbb{N}}$ given by the iteration $x^{k+1} = \phi(x^k), k \in \mathbb{N}$, for every $x^0 \in B(w, r)$.*

Proof. Because $B(w, r) \subset \mathbb{R}^n$ is a closed sphere in the space $\mathbb{R}^n$, will be a Banach space, too. Now we apply the Banach fixed point theorem for the function $\phi : B(w, r) \to B(w, r)$. □

We say that the function $\phi : B(w, r) \to \mathbb{R}^n$ is differentiable function on the closed sphere $B(w, r)$, if it is differentiable in every point $x \in B(w, r)$. If the point x is a boundary point of the closed sphere $B(w, r)$, then we suppose that the function ϕ is defined on a small open disc with center x and it is differentiable in x.

Theorem 4. *Let $\phi : B(w, r) \to \mathbb{R}^n$ be a continuously differentiable function on the closed sphere $B(w, r)$, and $x \in B(w, r), h \in \mathbb{R}^n$ such that $x + h \in B(w, r)$. If there exists $M \geq 0$ positive real number such that for every $u \in [x, x+h] = \{x+t\cdot h / t \in [0, 1]\}$ (the line segment $[x, x + h] \subset \mathbb{R}^n$ with endpoints x and $x + h$) we have $||D(\phi)(u)|| \leq M$, then $||\phi(x+h) - \phi(x)|| \leq M \cdot ||h||$.*

Proof. See for example [7]. □

Theorem 5. *If the euclidian norm of $D(\phi)(u)$ is bounded on the closed disc $B(w, r) \subset \mathbb{R}^n$, i.e. there exists $M > 0$ real number such that $||D(\phi)(u)|| \leq M$ for every $u \in B(w, r)$, then $||\phi(x) - \phi(y)|| \leq M \cdot ||x - y||$ for every $x, y \in B(w, r)$.*

Proof. It is immediately from the previous theorem 4. □

Theorem 6. *(general theorem for real n dimensional iterative functions) If $\phi : B(x^0, r) \to \mathbb{R}^n$ is a continuously differentiable function on the closed sphere $B(x^0, r), x^0 \in \mathbb{R}^n, r > 0$, such that the differential function $D(\phi)$ satisfies the inequality $0 \leq ||D(\phi)(u)|| \leq M < 1$ for every $u \in B(x^0, r)$ and the point $x^1 = \phi(x^0)$ verifies the inequality $||x^1 - x^0|| \leq (1 - M) \cdot r$, then:*

- *we can form the sequence $\{x^k\}_{k\in\mathbb{N}}$ with the iterative rule $x^{k+1} = \phi(x^k)$, $k \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ we have $x^k \in B(x^0, r)$;*
- *there exists the limit of the sequence $\{x^k\}_{k\in\mathbb{N}}$ and $\lim_{k\to\infty} x^k = x^* \in B(x^0, r)$;*
- *x^* is the unique solution of the equation $\phi(x) = x$ in the closed sphere $B(x^0, r)$.*

Proof. Using theorem 5 for every $x, y \in B(x^0, r)$ we get $||\phi(x) - \phi(y)|| \leq M \cdot ||x - y||$ with $M < 1$, so ϕ is a contraction. Also we obtain for every $x \in B(x^0, r)$ that $||\phi(x) - \phi(x^0)|| \leq M \cdot ||x - x^0||$. Now we show that $\phi(B(x^0, r)) \subset B(x^0, r)$, i.e. $\phi : B(x^0, r) \to B(x^0, r)$. Indeed, for every $x \in B(x^0, r)$ we get $||\phi(x) - x^0|| = ||\phi(x) - \phi(x^0) + x^1 - x^0|| \leq ||\phi(x) - \phi(x^0)|| + ||x^1 - x^0|| \leq M \cdot ||x - x^0|| + ||x^1 - x^0|| \leq M \cdot r + (1 - M) \cdot r = r$. Using theorem 3 we finish this proof. □

Let $f = (f_1, f_2, \ldots, f_n) : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function. We build the real n dimensional Steffensen's method in the following way: for every $i, j = \overline{1, n}$ let us consider the functions $h_{ij} : B(x^0, r) \to \mathbb{R}$,

$$h_{ij}(x) = \delta_{ij} \cdot \frac{f_i(x + f(x)) - f_i(x)}{f_{\varepsilon_i(j)}(x)},$$

where $\delta_{ij} \in \{0, 1\}$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are arbitrary permutations of the set $\{1, 2, \ldots, n\}$ and we define the matrix type function $h(x) = (h_{ij}(x))_{i,j=\overline{1,n}}$, where $f_j(x) \neq 0$ for every $j = \overline{1, n}$. Also we suppose that the determinant $det(h(x)) \neq 0$. Using the transformation of Steffensen's method, the system of equations $f(x) = 0$ is equivalent with the system of equations $x - (h(x))^{-1} \cdot f(x) = x$. We denote with $h^* : B(x^0, r) \to \mathbb{R}^n$ the function $h^*(x) = (h(x))^{-1} \cdot f(x)$. We can consider the iterative function $\phi(x) = x - (h(x))^{-1} \cdot f(x)$, the real n dimensional variant of the Steffensen's method. Now we apply theorem 6 for the iterative function ϕ and we obtain the following result for the function f: if the function $f : B(x^0, r) \to \mathbb{R}^n$ is a continuously differentiable function on $B(x^0, r)$, with $x^0 \in \mathbb{R}^n$ and $r > 0$, and $det(h(x)) \neq 0$ for every $x \in B(x^0, r)$, and $||1(x) - D(h^*)(x)|| \leq M < 1$ for every $x \in B(x^0, r)$ and $||h^*(x^0)|| = ||(h(x^0))^{-1} \cdot f(x^0)|| \leq (1 - M) \cdot r$, then the system of equations $f(x) = 0$ has a unique solution $x^* \in B(x^0, r)$ in the closed disc $B(x^0, r) \subset \mathbb{R}^n$ and x^* can be obtained as the limit of the sequence $\{x^k\}_{k\in\mathbb{N}}$, given by the iterative formula of the real n dimensional Steffensen's method: $x^{k+1} = x^k - (h(x^k))^{-1} \cdot f(x^k), k \in \mathbb{N}$.

3 Discussion and conclusion

Now if we fix all the constants $\delta_{ij} = 1$ for every $i, j = \overline{1, n}$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are the identical permutations, then we obtain the result in [6].

Next we consider the case $\delta_{ii} = 1$ for every $i = \overline{1, n}$, $\delta_{ij} = 0$ for all $i, j = \overline{1, n}$ with $i \neq j$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are the identical permutations. This means that

$$h_{ii}(x) = \frac{f_i(x + f(x)) - f_i(x)}{f_i(x)}$$

for $i = \overline{1, n}$ and $h_{ij}(x) = 0$ for all $i, j = \overline{1, n}$ with $i \neq j$. The matrix type function $h(x)$ has the diagonal elements

$$h_{ii}(x) = \frac{f_i(x + f(x)) - f_i(x)}{f_i(x)}$$

and the other elements there are equal zero. We calculate the invers matrix of the matrix $h(x)$ and we get an n dimensional matrix whose diagonal elements there are the expressions

$$\frac{f_i(x)}{f_i(x + f(x)) - f_i(x)}$$

and the other elements there are zero. We can conclude

$$\begin{aligned}\Phi(x) &= x - (h(x))^{-1} \cdot f(x) = \\ &= \left(x_1 - \frac{f_1^2(x)}{f_1(x + f(x)) - f_1(x)}, \right. \\ &\quad x_2 - \frac{f_2^2(x)}{f_2(x + f(x)) - f_2(x)}, \dots, \\ &\quad \left. x_n - \frac{f_n^2(x)}{f_n(x + f(x)) - f_n(x)} \right).\end{aligned}$$

If we denote

$$\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n) : D \subset \mathbb{R}^n \to \mathbb{R}^n,$$

then we have

$$\Phi_i(x) = x_i - \frac{f_i^2(x)}{f_i(x + f(x)) - f_i(x)}$$

for $i = \overline{1, n}$. We consider the euclidean norm on $\mathbb{R}^n$, and we calculate

$$\begin{aligned}\|x^1 - x^0\| &= \|\Phi(x^0) - x^0\| = \\ &= \left\{ \sum_{i=1}^{n} \left[\frac{f_i^2(x^0)}{f_i(x^0 + f(x^0)) - f_i(x^0)} \right]^2 \right\}^{1/2}.\end{aligned}$$

We can calculate the partial derivatives $(\Phi_i)'_{x_j}(x)$ for all $i, j = \overline{1, n}$, and all these partial derivatives depends on f. Next we consider

$$\|D\Phi(x)\| = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} [(\Phi_i)'_{x_j}(x)]^2 \right\}^{1/2}.$$

Now we apply theorem 6 for the iterative function Φ and we obtain the following result for the function f : if $f : B(x^0, r) \to \mathbb{R}^n$ is a continuously differentiable function on $B(x^0, r)$, where $x^0 \in \mathbb{R}^n$ and $r > 0$, and

$$\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[(\Phi_i)'_{x_j}(x) \right]^2 \right\}^{1/2} \leq M < 1$$

for every $x \in B(x^0, r)$, and

$$\left\{ \sum_{i=1}^{n} \left[\frac{f_i^2(x^0)}{f_i(x^0 + f(x^0)) - f_i(x^0)} \right]^2 \right\}^{1/2} \leq (1-M) \cdot r,$$

then the system of equations $f(x) = 0$ has a unique solution $x^* \in B(x^0, r)$ in the closed disc $B(x^0, r) \subset \mathbb{R}^n$ and x^* can be obtained like the limit of the sequence $\{x^k\}_{k \in \mathbb{N}}$, given by the iterative formula of the real n dimensional Steffensen's method:

$$x_i^{k+1} = x_i^k - \frac{f_i^2(x^k)}{f_i(x^k + f(x^k)) - f_i(x^k)},$$

where $i = \overline{1, n}$ and $k \in \mathbb{N}$.

If we choose other norms on the space $\mathbb{R}^n$, compatible with the euclidian norm, and we consider the corresponding dual, operator norms or compatible dual, operator norms, we can obtain other similar results.

We can see immediatly for the case $n = 1$ we reobtain the classical Steffensen's method.

In another paper we will show extensions of the chord method from the real line to he space $\mathbb{R}^n$.

References

[1] E. A. Volkov, Numerical Methods, Publishing Company MIR, Moscow, 1986.

[2] Béla Finta, Curs de analiză numerică, Universitatea "Petru Maior", Tg. Mures, 2004.

[3] Béla Finta, The Complex Version of a Result for Real Iterative Functions, Scientific Bulletin of the University of Medicine, Pharmacy, Sciences and Technology of Tirgu Mures, vol. 15(XXXII), no. 2, 2018, pag. 25–27.

[4] J. Cl. Evard and F. Jafari, A Complex Rolle's Theorem, Amer. Math. Monthly 99(1992), 858-861.

[5] Árpád Száz, A Cauchy Mean Value Theorem for Complex Functions, Math. Student 64(1995), 125–127.

[6] Béla Finta, A Generalization of a Result for Iterative Functions in $\mathbb{R}^n$, Acta Mariensis. Seria Technologica, Vol. 16 (XXXIII), no. 2, 2019, pag. 39–41.

[7] Wikipedia.

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