MINISTERUL EDUCAŢIEI NAŢIONALE
ROMÂNIA
UNIVERSITATEA DE MEDICINĂ, FARMACIE, ŞTIINŢE ŞI TEHNOLOGIE „GEORGE EMIL PALADE" DIN TÂRGU MUREŞ

sciendo

Acta Marisiensis. Seria Technologica

Vol. 20 (XXXIV) no. 2, 2023

10.2478/amset-2023-0016

NOTE ON: "THE COMPLEX VERSION OF A RESULT FOR REAL ITERATIVE FUNCTIONS"

Sushil Kumar BHUIYA[1], **Gopal DAS**[2],

[1] *Department of Mathematics, Krishnath College (Murshidabad University)*
1, Sahid Surya Sen Road, Gora Bazar, Berhampore, West Bengal, India 742101
[1]skbhuiyamu@gmail.com

[2]*Department of Mathematics, Murshidabad University*
1, Sahid Surya Sen Road, Gora Bazar, Berhampore, West Bengal, India 742101
[2]dasgopalmath2000@gmail.com

Abstract

Finta [2], recently proposed a complex version of iteration procedures for holomorphic functions. The general theorem of the complex iteration function has developed by using the complex mean value theorem and discussed several iterative procedures for holomorphic functions. In this paper, we redevelop the general theorem of the complex iteration function by applying the fundamental theorem of the complex line integral. It is shown that all the results derived in the paper of Finta have been improved by the results of this paper.

Keywords: Complex nonlinear equations, Complex iterative functions, Banach fixed point theorem, Fundamental theorem of the complex line integral

1. Introduction

Numerical methods for finding the root of real-valued functions have been an important topic to study. It is known the following result for iterative functions on the real line.

Theorem 1. (General Theorem for Real Iterative Functions [2, 8]) If φ is derivable on the interval J $= [x_0 - \delta,\ x_0 + \delta\ ]$, $\delta > 0$ and the derivative function φ' satisfies the inequality $0 \le |\varphi'(x)| \le m < 1$ for every $x \in$ J and the point $x_1 = \varphi(x_0)$ verifies the inequality $|x_1 - x_0| \le (1 - m)\delta$, then:

- we can form the sequence $\{x_k\}_{k\in\mathbb{N}}$ with the iterative rule $x_{k+1} = \varphi(x_k)$, $k \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ we have $x_k \in$ J;
- there exist x^* such that $\lim_{n\to\infty} x_k = x^* \in$ J;
- x^* is the unique solution of the equation $\varphi(x) = x$ in the interval J.

The general theorem for real iterative functions has extensively been applied to develop several root-finding methods for real-valued functions [3,4,6,7,10]. In the complex field, the holomorphic version of the mean value theorem was proposed by Evard and Jafari [1]. Finta [2] adopted the complex mean value to develop the iteration methods for complex-valued functions. Consequently, in [2], Finta proved the complex version of the theorem 1 by denoting the closed disc $B(w,r) = \{z \in \mathbb{C}: |z-w| \leq r\}$ in the complex plane $\mathbb{C}$, with centre w and radius $r > 0$ and using the following Banach fixed point theorem in the case of the closed disc $B(w,r)$:

Theorem 2. Let $\varphi : B(w,r) \to B(w,r)$ be a contraction, i.e., there exists the constant $\alpha \in [0,1)$ such that $|\varphi(z) - \varphi(v)| \leq \alpha \cdot |z - v|$ for every $z, v \in B(w,r)$. Then the function φ has a unique fixed point in $B(w,r)$, which can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ given by the iteration $z_{k+1} = \varphi(z_k)$, $k \in \mathbb{N}$, for every $z_0 \in B(w,r)$ [2].

Proof: Because $B(w,r) \subset \mathbb{C}$ is a closed disc in the complex plane $\mathbb{C}$, will be a Banach space, too. Now, we apply the Banach fixed point theorem for the function $\varphi: B(w,r) \to B(w,r)$. Hence, the proof.

Theorem 3. (General Theorem for Complex Iterative Functions [2]) If $\varphi : B(z_0,r) \to \mathbb{C}$ is a holomorphic function on the closed disc $B(z_0,r)$, $z_0 \in \mathbb{C}$, $r > 0$, such that the derivative function φ' satisfies the inequality $0 \leq |\varphi'(z)| \leq m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0,r)$ and the point $z_1 = \varphi(z_0)$ verifies the inequality $|z_1 - z_0| \leq (1 - \sqrt{2} \cdot m) \cdot r$, then:

- we can form the sequence $\{z_k\}_{k\in\mathbb{N}}$ with the iterative rule $z_{k+1} = \varphi(z_k)$, $k \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ we have $z_k \in B(z_0,r)$;
- there exists the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ and $\lim\limits_{n\to\infty} z_k = z^* \in B(z_0,r)$;
- z^* is the unique solution of the equation $\varphi(z) = z$ in the closed disc $B(z_0,r)$.

The purpose of this study is to improve the above theorem by using the fundamental theorem of line integration. Consequently, we obtain some important iterative procedures in the complex domain.

2. Main Result

In this section, we redevelop the general theorem for Complex iterative function by using the fundamental theorem of line integral. First of all, we recall the definition of holomorphic function. We say that the function $\varphi : B(z_0,r) \to \mathbb{C}$ is holomorphic function on the closed disc $B(w,r)$, if it is complex derivable in every complex point $z \in B(w,r)$. If the point z is a boundary point of the closed disc $B(w,r)$, then we suppose that the complex function φ is defined on a small open disc with centre z and it is complex derivable in z.

Theorem 4. (Fundamental Theorem of Complex Line Integrals [5])

If $f(z)$ is a complex holomorphic function on an open region A and γ is a curve in A from z_0 to z_1 then

$$\int_\gamma f'(z)dz = f(z_1) - f(z_0).$$

Theorem 5. (Improved General Theorem for Complex Iterative Functions) If $\varphi : B(z_0,r) \to \mathbb{C}$ is a holomorphic function on the closed disc $B(z_0,r)$, $z_0 \in \mathbb{C}$, $r > 0$, such that the derivative function φ' satisfies the inequality $0 \leq |\varphi'(z)| \leq m < 1$ for every $z \in B(z_0,r)$ and the point $z_1 = \varphi(z_0)$ verifies the inequality $|z_1 - z_0| \leq (1 - m) \cdot r$, then:

- we can form the sequence $\{z_k\}_{k\in\mathbb{N}}$ with the iterative rule $z_{k+1} = \varphi(z_k)$, $k \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ we have $z_k \in B(z_0,r)$;
- there exists the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ and $\lim\limits_{n\to\infty} z_k = z^* \in B(z_0,r)$;

- z^* is the unique solution of the equation $\varphi(z) = z$ in the closed disc $B(z_0, r)$.

Proof: For any $z \in B(z_0, r)$, we get the following by applying the fundamental theorem of line integration. Let γ be the line segment between z and z_0, then

$$|\varphi(z) - \varphi(z_0)| = |\oint_\gamma \varphi'(z)dz|$$

$$\leq \oint_\gamma |\varphi'(z)||dz|$$

$$\leq \oint_\gamma m|\frac{dz}{dt}|dt$$

$$[\because |\varphi'(z)| \leq m]$$

$$\leq \int_0^1 m|z - z_0|dt$$

$$[\because z(t) = z_1 + (z_2 - z_1)t, t \in [0, 1]]$$

$$= m|z - z_0|.$$

Therefore $|\varphi(z) - \varphi(z_0)| \leq m|z - z_0|$. Now, we show that $\varphi(B(z_0, r) \subset B(z_0, r)$, i.e., $\varphi: B(z_0, r) \to B(z_0, r)$. Indeed, for every $z \in B(z_0, r)$ we obtain:

$$|\varphi(z) - z_0| = |\varphi(z) - \varphi(z_0) + z_1 - z_0|$$

$$\leq |\varphi(z) - \varphi(z_0)| + |z_1 - z_0|$$

$$\leq m|z - z_0| + |z_1 - z_0|$$

$$\leq m \cdot r + (1 - m) \cdot r$$

$$= r.$$

Again, for every $z, v \in B(z_0, r)$, $z \neq v$, we get $|\varphi(z) - \varphi(v)| \leq m|z - v|$. We choose $\alpha = m < 1$ and the complex function $\varphi : B(z_0, r) \to B(z_0, r)$satisfies the conditions of theorem 2. Consequently, we can deduce the statements of theorem 5. Hence, this completes the proof.

Observation 1. It is clear that the result which we find by applying the fundamental theorem of line integration is better than the result obtained by the theorem 3.

3. Discussion

In this section, we discuss some applications of the result which is obtained in theorem 5. It is noted that all results are improvement of the results derived in [2].

Result 1. Using the translation, the complex equation $f(z) = 0$ is equivalent with the complex equation $z + f(z) = z$. We can consider the complex iterative function $\varphi(z) = z + f(z)$. Now we apply theorem 5 for the iterative function φ and we obtain the following result for the function f if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$, $r > 0$, $| 1 + f'(z) | \leq m < 1$ for every $z \in B(z_0, r)$, and $|f(z_0)| \leq (1 - m) \cdot r$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula $z_{k+1} = z_k + f(z_k), k \in \mathbb{N}$.

Result 2. Using the translation, the complex equation $f(z) = 0$ is equivalent with the complex equation $z + \omega \cdot f(z) = z, \omega \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We can consider the complex iterative function $\varphi(z) = z + \omega \cdot f(z)$, $\omega \in \mathbb{C}^*$. Now we apply the theorem 5 for the iterative function φ and we obtain the following result for the function f if the complex function $f: B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$, $r > 0$, $|1 + \omega \cdot f'(z)| \leq m < 1$ for every $z \in B(z_0, r)$, and $|f(z_0)| \leq (1 - m) \cdot \frac{r}{|\omega|}$, then the complex equation $f(z) = 0$ has a unique solution z^* in the the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ given by the iterative formula $z_{k+1} = z_k + \omega \cdot f(z_k), k \in \mathbb{N}$.

Result 3. Using the complex Newton's transformation, the complex equation $f(z) = 0$ is equivalent with the complex equation $z - \frac{f(z)}{f'(z)} = z$, where we suppose that $f'(z) \neq 0$. We can consider the complex iterative function $(z) = z - \frac{f(z)}{f'(z)}$, the complex variant of the Newton's method. Now we apply theorem 5 for the iterative function φ and we obtain the following result for the function f if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$, $r > 0$, and $f'(z) \neq 0$ for every $z \in B(z_0, r)$, and $\frac{|f(z) f''(z)|}{[f'(z)]^2} \leq m < 1$ for every $z \in B(z_0, r)$, and $|\frac{f(z_0)}{f'(z_0)}| \leq (1-m) \cdot r$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k \in \mathbb{N}}$, given by the complex Newton's iterative formula $z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}, k \in \mathbb{N}$.

Result 4. Using the transformation of complex parallel method, the complex equation $f(z) = 0$ is equivalent with the complex equation $z - \frac{1}{\lambda} \cdot f(z) = z$, $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We can consider the complex iterative function $\varphi(z) = z - \frac{1}{\lambda} \cdot f(z)$, $\lambda \in \mathbb{C}^*$, the complex variant of the parallel method. Now we apply theorem 5 for the iterative function φ and we obtain the following result for the function f if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$, and $r > 0$, $|1 - \frac{1}{\lambda} \cdot f'(z)| \leq m < 1$ for every $z \in B(z_0, r)$, and $|f(z_0)| \leq (1-m) \cdot r \cdot |\lambda|$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k \in \mathbb{N}}$, given by the iterative formula of the complex parallel method $z_{k+1} = z_k - \frac{1}{\lambda} \cdot f(z_k), k \in \mathbb{N}$.

Result 5. Using the transformation of complex chord method the complex equation $f(z) = 0$ is equivalent with the complex equation $z - f(z) \cdot \frac{z-a}{f(z)-f(a)} = z$, where $a \in \mathbb{C}$ is a fixed complex number $, z \neq a$ and $f(z) \neq f(a)$ for $z \neq a$.We can consider the complex iterative function $\varphi(z) = z - f(z) \cdot \frac{z-a}{f(z)-f(a)}$, where $a,\ z \in \mathbb{C}, z \neq a$ and $f(z) \neq f(a)$ for $z \neq a$, the complex variant of the chord method. The condition $f(a) \neq 0$ implies for $z \neq a$ that $\varphi(z) \neq a$, too. Indeed, from equality $\varphi(z) = a$, we get $z - f(z) \cdot \frac{z-a}{f(z)-f(a)} = a$, which means that $(z - a) \cdot \frac{f(a)}{f(z)-f(a)} = 0$. Consequently, from the condition $f(a) \neq 0$ we can deduce $z = a$. Now we apply theorem 5 for the iterative function φ and we obtain the following result for the function f if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$, and $r > 0$and $a \in B(z_0, r)$ is a fixed complex number such that $f(a) \neq 0$ and for every $z \in B(z_0, r) - \{a\}$ we have $f(z) \neq f(a)$, $|\frac{|f(a)||f(a) - f(z) + f'(z) \cdot (z-a)|}{|f(z) - f(a)|^2}| \leq m < 1$ for every $z \in B(z_0, r) - \{a\}$, and $|\frac{(z_0 - a) f(z_0)}{f(z_0) - f(a)}| \leq (1-m) \cdot r$ with $z_0 \neq a$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k \in \mathbb{N}}$, given by the iterative formula of the complex chord method $z_{k+1} = z_k - f(z_k) \cdot \frac{z_k - a}{f(z_k) - f(a)},\ k \in \mathbb{N}$.

Result 6. Using the transformation of complex Steffensen's method, the complex equation $f(z) = 0$ is equivalent with the complex equation $z - \frac{f^2(z)}{f(z+f(z)) - f(z)} = z$. We can consider the complex iterative function $\varphi(z) = z - \frac{f^2(z)}{f(z+f(z)) - f(z)}$, the complex variant of the Steffensen's method. Now we apply theorem 5 for the iterative function φ and we obtain the following result for the function f if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$, and $r > 0$, $|1 - \frac{\{2 \cdot f(z) \cdot f'(z) \cdot [f(z+f(z)) - f(z)]\} - \{f^2(z) \cdot [f'(z+f(z)) \cdot (1 + f'(z)) - f'(z)]\}}{[f(z+f(z)) - f(z)]^2}| \leq m < 1$ for every $z \in B(z_0, r)$, and $|\frac{f^2(z_0)}{f(z_0 + f(z_0)) - f(z_0)}| \leq (1-m) \cdot r$, then the complex equation $f(z)=0$ has a unique solution z^* in the closed disc $B(z_0,r)\subset\mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula of the complex parallel method $z_{k+1}=z_k-\frac{f^2(z_k)}{f(z_k+f(z_k))-f(z_k)}, k\in\mathbb{N}$.

Result 7. The equation $z^n-a=0$, where $a\in\mathbb{C}^*$ is a fixed complex number and $n\in\mathbb{N}$, $n\geq 2$ is a fixed natural number, is equivalent with the complex equation $\frac{1}{2}\cdot(z+\frac{a}{z^{n-1}})=z$, $z\in\mathbb{C}^*=\mathbb{C}-\{0\}$. We can consider the complex iterative function $\varphi(z)=\frac{1}{2}\cdot(z+\frac{a}{z^{n-1}})$ .Now we apply theorem 5 for the iterative function φ and we obtain the following result : if we fix $z_0\in\mathbb{C}$ and $r>0$ such that $0\notin B(z_0,r)$ and $\left|1-\frac{(n-1)a}{z^n}\right|\leq m<1$ for every $z\in B(z_0,r)$, and $|z_0-\frac{a}{z_0^{n-1}}|\leq(1-m)\cdot r$. Then the complex sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula $z_{k+1}=\frac{1}{2}\cdot(z_k+\frac{a}{z_k^{n-1}})$ is convergent and tends to $\sqrt[n]{a}$, the complex n^{th} root of $a\in\mathbb{C}^*$.We mention the particular case for $n=2$: if we fix $z_0\in\mathbb{C}$, and $r>0$ such that $0\notin B(z_0,r)$, and $|1-\frac{a}{z^2}|\leq m<1$ for every $z\in B(z_0,r)$, and $|z_0-\frac{a}{z_0}|\leq(1-m)\cdot r$, then the complex sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula $z_{k+1}=\frac{1}{2}\cdot(z_k+\frac{a}{z_k})$ is convergent and tends to $\sqrt{a}$, the complex square root of $a\in\mathbb{C}^*$.

4. Conclusion

In this paper, we have considered the complex version of a result for real iterative functions. We have reconstructed the general theorem of the complex iteration function by applying the fundamental theorem of the complex line integral to improve the existing results. Moreover, several numerical schemes form complex valued functions have been updated by using our reconstructed theorem. For future research on this work, it would be interesting to apply the proposed concepts in the complex conformable functions [9].

5. References

[1] Evard J. Cl. and Jafari F., A Complex Roll's Theorem, Amer. Math Monthly 99(1992), 858-861.

[2] Finta, B., The complex version of a result for real iterative functions., Scientific Bulletin of the University of Medicine, Pharmacy, Sciences and Technology of Tirgu Mures 15(2018): 25-27.

[3] Finta, B., A Generalization of a Result for Iterative Functions in R^n., Acta Marisiensis., Seria Technologica 16, no. 2 (2019): 39-41.

[4] Homeier, H. H. H., A modified Newton method for root finding with cubic convergence., Journal of Computational and Applied Mathematics 157, no. 1 (2003): 227-230.

[5] Ponnusamy S. and Silverman H., Complex variables with applications. Springer Science and Business Media, 2007.

[6] Rhoades, B. E., Some fixed-point iteration procedures., International Journal of Mathematics and Mathematical Sciences 14 (1991): 1-16.

[7] Sabharwal, C. L., Blended root finding algorithm outperforms bisection and regula falsi algorithms., Mathematics 7, no. 11 (2019): 1118.

[8] Volcov E. A., Numerical Methods, Publishing Company MIR, Moscow, 1986.

[9] Uçar, S., and Nihal Ö., Complex conformable Rolle's and Mean Value Theorems., Mathematical Sciences 14, no. 3 (2020): 215-218.

[10] Zheng, Q., Zhao, P., Zhang, L., and Ma, W., Variants of Steffensen-secant method and applications. Applied mathematics and computation 216, no. 12 (2010): 3486-3496.