MINISTERUL EDUCAŢIEI NAŢIONALE
ROMÂNIA
UNIVERSITATEA DE MEDICINĂ, FARMACIE, ŞTIINŢE ŞI TEHNOLOGIE „GEORGE EMIL PALADE" DIN TÂRGU MUREŞ

Acta Marisiensis. Seria Technologica
Vol. 21 (XXXVIII) no. 1, 2024
ISSN 2668-4217, ISSN-L 2668-4217

FINITE NUMBER SOLUTIONS OF A DIOPHANTINE ALGEBRAIC SYSTEM

Béla FINTA
George Emil Palade University of Medicine, Pharmacy, Science, and Technology of Târgu Mureş, Gheorghe Marinescu Street, no. 38, 540139 Târgu Mureş, Romania
email: bela.finta@umfst.ro

Abstract

The purpose of this paper is to demonstrate that a Diophantine algebraic system has finite number solutions.

Key words: Diophantine algebraic system, Diophantine algebraic equation

1 Introduction

In [1] we can find the following open question: determine all $x_k, y_k \in \mathbb{N}$ such that

$$\begin{cases} x_1 + x_2 + x_3 = y_1 \cdot y_2 \cdot y_3 \\ y_1 + y_2 + y_3 = x_1 \cdot x_2 \cdot x_3. \end{cases} \tag{1}$$

In [2] we showed that this system has finite number solutions in the set of natural numbers and we solved this system in algebraic way by hand calculus and we obtained 97 different solutions in the set of natural numbers:

$$(x_1, x_2, x_3, y_1, y_2, y_3) \in \{(0,0,0,0,0,0); (1,2,5,1,1,8); (1,1,8,1,2,5); (1,3,3,1,1,7); (1,1,7,1,3,3); (2,2,2,1,1,6); (1,1,6,2,2,2), (1,2,3,1,2,3)\}.$$

In order to obtain all solutions of the system (1) we permute the values x_i, $i = 1,2,3$ and separatelly the values y_i, $i = 1,2,3$, after we combine every permutation of x_i, $i = 1,2,3$ with every permutation of y_i, $i = 1,2,3$ and in this way finally we can deduce all 97 different solutions of (1) in the set of natural numbers.

Now we announce a direct generalization of this system: for every fixed $n, m \in \mathbb{N} \setminus \{0\}$ determine all $x_k, y_l \in \mathbb{N}$, $k = \overline{1,n}$, $l = \overline{1,m}$ such that

$$\begin{cases} x_1 + x_2 + \ldots + x_n = y_1 \cdot y_2 \cdot \ldots \cdot y_m \\ y_1 + y_2 + \ldots + y_m = x_1 \cdot x_2 \cdot \ldots \cdot x_n. \end{cases} \tag{2}$$

We solved this system by hand calculus for $n, m \in \{1,2,3\}$ in [3]:

1. if $n = m = 1$, then we obtain the trivial system $x_1 = y_1 \in \mathbb{N}$;
2. if $n = 1, m = 2$, then we find two solutions $(x_1, y_1, y_2) \in \{(0,0,0)\}, (4,2,2)\}$;
3. if $n = 1, m = 3$, then we find seven different solutions of the system:

   $$(x_1, y_1, y_2, y_3) \in \{(0,0,0,0)\}, (6,1,2,3), (6,1,3,2), (6,2,1,3), (6,2,3,1), (6,3,1,2), (6,3,2,1)\};$$

4. if $n = 2, m = 1$ is similar with the case 2;
5. if $n = m = 2$ we find ten different solutions:

$$(x_1, x_2, y_1, y_2) \in \{(0,0,0,0)\}, (2,3,1,5), (2,3,5,1), (3,2,1,5), (3,2,5,1), (1,5,2,3), (1,5,3,2), (5,1,2,3), (5,1,3,2), (2,2,2,2)\};$$

6. if $n = 2, m = 3$ we have nineteen different solutions of the system:

$$(x_1, x_2, y_1, y_2, y_3) \in \{(0,0,0,0,0)\}, (1,7,1,2,4), (1,7,1,4,2), (1,7,2,1,4), (1,7,2,4,1), (1,7,4,1,2), (1,7,4,2,1), (7,1,1,2,4), (7,1,1,4,2), (7,1,2,1,4), (7,1,2,4,1), (7,1,4,1,2), (7,1,4,2,1), (2,4,1,1,6), (2,4,1,6,1), (2,4,6,1,1), (4,2,1,1,6), (4,2,1,6,1), (4,2,6,1,1)\};$$

7. if $n = 3, m = 1$ is similar with the case 3;
8. if $n = 3, m = 2$ is similar with the case 6;
9. if $n = 3, m = 3$ we have 97 different solutions presented above.

2 Main part

First of all we present the following results: let us consider the Diophantine algebraic equation in the set of natural numbers:

$$x_1 + x_2 + \ldots + x_n = A \cdot x_1 \cdot x_2 \ldots x_n \qquad (3)$$

where $x_i \in \mathbb{N}$ for every $i = \overline{1,n}$ are the unknowns and $A \in \mathbb{N}$ is a fixed natural number.

Proposition 1. *The equation* (3) *has finite number solutions in the set of natural numbers for $n \geq 2$.*

Proof. If $A = 0$, then $x_1 + x_2 + \ldots + x_n = 0$ with the banal solution $x_1 = x_2 = \ldots = x_n = 0$.

Next let be $A \geq 1$. If there exists $i = \overline{1,n}$ such that $x_i = 0$ then we obtain $x_1 + x_2 + \ldots + x_n = 0$ with the banal solution $x_1 = x_2 = \ldots = x_n = 0$. So we can suppose for every $i = \overline{1,n}$, that $x_i \neq 0$. This means $x_i \geq 1$ for every $i = \overline{1,n}$. Without loss of generality we can suppose $1 \leq x_1 \leq x_2 \ldots \leq x_n$. By permutation we can obtain all solutions of the Diophantine algebraic equation (3).

We mention the case $n = 1$ and from (3) we get the equation $x_1 = A \cdot x_1$. If $A = 1$, then we have infinite solutions and if $A \geq 2$ we receive one root $x_1 = 0$.

Next let be $n \geq 2$. We have two cases:

a) From $x_{n-1} \leq n$ we deduce $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq n$. We can observe that there exist finite number selections of the unknowns $x_1, x_2, \ldots, x_{n-1}$ such that $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq n$.

For a fixed selection of the unknowns $x_1, x_2, \ldots, x_{n-1}$ such that $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq n$ from (3) we obtain an equation in the unknown x_n. We can see immediately that this equation in the unknown x_n is an algebraic equation of the first degree. Indeed,

$$x_1 + x_2 + \ldots + x_{n-1} = x_n \cdot (A \cdot x_1 \cdot x_2 \ldots x_{n-1} - 1).$$

If $A \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} = 1$ then $A = x_1 = x_2 = \ldots = x_{n-1} = 1$ so $x_1 + x_2 + \ldots + x_{n-1} = n - 1$, i.e. $n - 1 = 0$, which is a contradiction with $n \geq 2$. If $A \cdot x_1 \cdot x_1 \cdot \ldots \cdot x_n \neq 1$, then for x_n we obtain one rational solution. We have at most one solution for x_n in the set of natural numbers. Finally we can conclude that in this case our Diophantine equation (3) has finite number solutions in the set of natural numbers.

b) If $x_{n-1} > n$ results $x_{n-1} \geq n + 1$. We divide the Diophantine equation (3) with $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ and we get the form

$$\begin{aligned} A = &\frac{1}{x_2 \cdot x_3 \cdot \ldots \cdot x_n} + \frac{1}{x_1 \cdot x_3 \cdot \ldots \cdot x_n} + \ldots + \\ &+ \frac{1}{x_1 \cdot x_2 \cdot \ldots \cdot x_{n-3} \cdot x_{n-1} \cdot \cdot x_n} + \\ &+ \frac{1}{x_1 \cdot x_2 \cdot \ldots \cdot x_{n-2} \cdot x_n} + \\ &+ \frac{1}{x_1 \cdot x_2 \cdot \ldots \cdot x_{n-3} \cdot x_{n-2} \cdot x_{n-1}}. \end{aligned}$$

But in this case $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-2}$ and $n+1 \leq x_{n-1} \leq x_n$, so we can majorize the right side of the equation from above taking possible minimal values for the unknowns:

$$\begin{aligned} 1 \leq A &\leq \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+1)^2} \\ &+ \frac{1}{n+1} + \frac{1}{n+1} = \frac{n-2}{(n+1)^2} + \frac{2}{n+1} \\ &= \frac{n-2+2(n+1)}{(n+1)^2} = \frac{3n}{(n+1)^2} < 1, \end{aligned}$$

because $3n < (n+1)^2$, which is equivalent with the inequality $n \cdot (n-1) + 1 > 0$.

This is true for $n \geq 2$ and means we do not have solutions.

Consequently our Diophantine algebraic equation (3) has finite number solutions in the set of natural numbers. □

Remark 1. *If $A = 1$ then from proposition 1 we obtain the equation $x_1 + x_2 + \ldots + x_n = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, which has finite number solutions in the set of natural numbers for $n \geq 2$.*

Remark 2. *For $A = 1$ and $n \geq 2$ fixed natural number such that $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-2} \leq x_{n-1}$ and $n + 1 \leq x_{n-1} \leq x_n$ we get $x_1 + x_2 + \ldots + x_n < x_1 \cdot x_2 \cdot \ldots \cdot x_n$.*

Remark 3. *For $A = 1$ and $n = 3$ we obtain the problem $x_1 + x_2 + x_3 = x_1x_2x_3$ proposed for mathematical olimpiad [4] and we have 7 solutions $(x_1, x_2, x_3) \in \{(0, 0, 0), (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$.*

Remark 4. *For $A = 1$ and $n = 4$ we obtain the Diophantine equation $x_1+x_2+x_3+x_3 = x_1\cdot x_2\cdot x_3\cdot x_4$ and we solved in [5] by hand calculus and we get 13 different solutions in the set of natural numbers:*

$$(x_1, x_2, x_3, x_4) \in \{(0, 0, 0, 0), (2, 4, 1, 1), (2, 1, 4, 1), (2, 1, 1, 4), (4, 2, 1, 1), (4, 1, 2, 1), (4, 1, 1, 2), (1, 2, 4, 1), (1, 2, 1, 4), (1, 4, 1, 2), (1, 4, 2, 1), (1, 1, 2, 4), (1, 1, 4, 2)\}.$$

Next we generalise proposition 1.

Let us consider the following Diophantine algebraic equation in the set of natural numbers:

$$x_1 + x_2 + \ldots + x_n + B = A \cdot x_1 \cdot x_2 \ldots x_n \tag{4}$$

where $x_i \in \mathbb{N}$ for every $i = \overline{1, n}$ are the unknowns and $A, B \in \mathbb{N}$ are fixed natural numbers.

Proposition 2. *The equation* (4) *has finite number solutions in the set of natural numbers for $n \geq 2$.*

Proof. I. If $A = 0$, then $x_1 + x_2 + \ldots + x_n + B = 0$. If $B = 0$ then we obtain the banal solution $x_1 = x_2 = \ldots = x_n = 0$. If $B \geq 1$ then we do not have solutions.

II. Next let be $A \geq 1$. If there exists $i = \overline{1, n}$ such that $x_i = 0$ we obtain $x_1 + x_2 + \ldots + x_n + B = 0$. If $B = 0$ then we obtain the banal solution $x_1 = x_2 = \ldots = x_n = 0$. If $B \geq 1$ then we do not have solutions. So we can suppose for every $i = \overline{1, n}$ that $x_i \neq 0$. This means $x_i \geq 1$ for every $i = \overline{1, n}$. Without loss of generality we can suppose $1 \leq x_1 \leq x_2 \leq \ldots \leq x_n$. By permutation we can obtain all solutions of the Diophantine algebraic equation (4).

We have the following subcases when $A \geq 1$:

II.1. If $B = 0$ from equation (4) we reobtain the equation (3). Using proposition 1 we can deduce that our equation has finite number solutions in the set of natural numbers for $n \geq 2$.

II.2. If $B \geq 1$ we make the discussion corresponding to n.

II.2.a. In the first case $n = 1$ and from (4) we get the equation $x_1 + B = A \cdot x_1$. If $A = 1$ and $B = 0$ we have infinite solutions. If $A = 1$ and $B \geq 1$ we do not have solutions. If $A \geq 2$ them from the equation $x_1 + B = A \cdot x_1$ we obtain $x_1 = \frac{B}{A-1}$, so we have at most one natural number solution.

II.2.b. In the second case $n \geq 2$ and the equation (4) we put in the form

$$\begin{aligned} &1 + 1 + \ldots + 1 + x_1 + x_2 + \ldots + x_n \\ &= A \cdot 1 \cdot 1 \cdot \ldots \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n, \end{aligned}$$

where the number 1 appears B times in the above equation. We denote $z_1 = z_2 = \ldots = z_B = 1$, $z_{B+1} = x_1$, $z_{B+2} = x_2, \ldots,$ $z_{B+n} = x_n$, and we obtain the equation

$$\begin{aligned} &z_1 + z_2 + \ldots + z_B + z_{B+1} + z_{B+2} + \ldots + z_{B+n} \\ &= A \cdot z_1 \cdot z_2 \cdot \ldots \cdot z_B \cdot z_{B+1} \cdot z_{B+2} \cdot \ldots \cdot z_{B+n}, \end{aligned}$$

where

$$\begin{aligned} &1 \leq z_1 \leq z_2 \leq \ldots \leq z_B \leq z_{B+1} \leq z_{B+2} \\ &\leq \ldots \leq z_{B+n}. \end{aligned}$$

Using proposition 1 we have finite number solutions in the set of natural numbers. □

Remark 5. *For $B = 0$ from proposition 2 we reobtain proposition 1.*

Remark 6. *For $A = B = 3$ I proposed the problem $x_1 + x_2 + x_3 + 3 = 3 \cdot x_1 \cdot x_2 \cdot x_3$ for mathematical competition. This Diophantine equation does not have solution in the set of natural numbers.*

3 Conclusions

We finish our work with the following proposition.

Proposition 3. *The Diophantine system* (2) *has finite number solutions in the set of natural numbers for $n, m \in \mathbb{N} \setminus \{0\}$ excepting $n = m = 1$.*

Proof. Now let us consider the system (2) for the case $n = 1$ and $m \geq 2$

$$\begin{cases} x_1 = y_1 \cdot y_2 \cdot \ldots \cdot y_m \\ y_1 + y_2 + \ldots + y_m = x_1, \end{cases} \tag{5}$$

so $y_1 + y_2 + \ldots + y_m = y_1 \cdot y_2 \cdot \ldots \cdot y_m$.

Using remark 1 we have finite number solutions in the set of natural numbers. Analogously for the case $m = 1$ and $n \geq 2$.

Remain to verify the system (2) when $n \geq 2$ and $m \geq 2$.

If $x_{n-1} \leq n$ we eliminate from the system (2) the unknown x_n :

$$x_n = y_1 \cdot y_2 \cdot \ldots \cdot y_m - (x_1 + x_2 + \ldots + x_{n-1}),$$

so

$$\begin{aligned} &y_1 + y_2 + \ldots + y_m = x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} \cdot \\ &\cdot [y_1 \cdot y_2 \cdot \ldots \cdot y_m - (x_1 + x_2 + \ldots + x_{n-1})]. \end{aligned}$$

Let us denote $A = x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1}$ and $B = x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} \cdot (x_1 + x_2 + \ldots + x_{n-1})$. Using $1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq n$ means that we have finite number possibilities for the values A and B. We get

$$y_1 + y_2 + \ldots + y_m + B = A \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m.$$

Using proposition 2 we obtain finite number solutions for this equation in variables $y_1, y_2, \ldots, y_m$.

Consequently the unknown x_n will take finite number natural values.

Analoguesly, when $y_{m-1} \leq m$.

So the system (2) has finite number solution for $x_{n-1} \leq n$ or $y_{m-1} \leq m$. Remains the case, when $x_{n-1} \geq n+1$ and $y_{m-1} \geq m+1$. From $x_{n-1} \geq n+1$ and $y_{m-1} \geq m+1$ we deduce using remark 2 $x_1 + x_2 + \ldots + x_n < x_1 \cdot x_2 \cdot \ldots \cdot x_n$ and $y_1 + y_2 + \ldots + y_m < y_1 \cdot y_2 \cdot \ldots \cdot y_m$. Hence $x_1 \cdot x_2 \cdot \ldots \cdot x_n > x_1 + x_2 + \ldots + x_n = y_1 \cdot y_2 \cdot \ldots \cdot y_m > y_1 + y_2 + \ldots + y_m$, which gives a contradiction. □

Acknowledgement

The research of Béla Finta was performed in the frame of the Research Center on Artificial Intelligence, Data Science and Smart Engineering (Artemis).

References

[1] Bencze, M. (2010). *Open Question OQ. 3664*, Octogon Mathematical Magazine, vol. 18, no. 1, April 2010, Braşov, Romania, pp. 257, Print: ISSN 1222-5657, Online: ISSN 2248-1893.

[2] Finta, B. (2014). *Solution of the Open Question OQ. 3664 Proposed by M. Bencze*, Octogon Mathematical Magazine, vol. 22, no. 2, October 2014, Braşov, Romania, pp. 1116-1125, Print: ISSN 1222-5657, Online: ISSN 2248-1893.

[3] Finta, B. (2014). *Generalization of the Open Question OQ. 3664 Proposed by M. Bencze*, Virginia Marinescu, Mihály Bencze, The International Conference Mathematical Education in the Current European Context, The 12th of December 2014, Braşov, Romania, pp. 119–125, ISSN 2360-5324, ISSN-L 2360-5324.

[4] Gazeta Matematica, Supliment, A 67-a Olimpiadă Naţională de Matematică, Târgu Mureş, Sovata, 19-23 aprilie 2016, Societatea de Ştiinţe Matematice din România, pp. 5, ISSN 1584-9333.

[5] Finta, B. (2015). *About a Diophantine Algebraic Equation*, Virginia Marinescu, Mihály Bencze, The International Conference Mathematical Education in the Current European Context, The 11th of December 2015, Braşov, Romania, pp. 111–116, ISSN 2360-5324, ISSN-L 2360-5324.